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The Structure of Contours for non-degenerate Surfaces defined on a 2-dimensional Manifold. (**)

The main objective of this paper is to show that for a non-degenerate surface defined on a 2-dimensional manifold, there exists a representation for which almost all contours are arcs, points, or simple closed curves.

The definitions and notation used in this paper are either the same or equivalent to those used by ROBERT E. FULLERTON [1] and LAMBERTO CESARI [2].

Let M be a 2-dimensional manifold and let T be a continuous mapping from M into n-dimensional Euclidean space E_n. Zhe mapping T defines a FRÉCHET surface S. Let $[S]$ denote the set of points in E_n occupied by the surface. On $[S]$ define a real valued function f with upper and lower bounds t_1 and t_2, respectively. The *contour* $C(t)$ associated with f, T, t is the set of all points $p \in M$ such that $f[T(p)] = t$, for $t \in [t_1, t_2]$; i.e.

$$C(t) = \{ p: p \in M \text{ and } f[T(p)] = t \text{ for } t \in [t_1, t_2] \}.$$

Let $\quad D^{-}(t) = \{ p: p \in M \text{ and } f[T(p)] < t \text{ for } t \in [t_1, t_2] \}.$

Let $\quad D^{+}(t) = \{ p: p \in M \text{ and } f[T(p)] > t \text{ for } t \in [t_1, t_2] \}.$

Let $\alpha(t)$ be a component of $D^{-}(t)$ and let γ be a component of $\alpha^{*}(t) - \alpha(t)$, where by * we denote the boundary of the set. The set $A = A(\alpha, \gamma)$ is the set

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(**) This is a generalization of ROBERT E. FULLERTON's paper: *The Structure of Contours of a Fréchet Surface.* I wish to express my sincere thanks to Dr. FULLERTON for his kind and devoted advice.

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of points of M which are in α plus those which are separated from γ by other components of $\alpha^*(t) - \alpha(t)$ plus those components of $\alpha^*(t) - \alpha(t)$ which separate points of Q from γ.

The contours may be in general of a rather complex topological nature, and it is desirable in some cases to substitute for the contour a somewhat simpler smoothed contour whose image is less than (or equal) to the original contour in lenght and has simpler properties. R. E. FULLERTON discusses 2 smoothing methods in [1]. Since the first of his method will be applied in the proof of this theorem, it will be described here briefly:

Denote by $\Gamma \subset M$ the set of all maximal continua of constancy for T in M and denote by $\sigma' \subset \Gamma$ the set of all such continua which intersect γ. Consider the set $\{w\}_{(A,\gamma)}$ of all prime ends of A corresponding to points on γ. This set can be ordered [2]. For $w_1 < w_2 \in \{w\}_{(A,\gamma)}$ let w', w'' be such that $w_1 \leqslant w' < w'' \leqslant w_2$. Let E_w denote the points of γ associated with a prime end w. Assume that $E_{w'} \cap E_{w''} \neq 0$. For any end w''' with $w' < w''' < w''$ let $\sigma'(w', w'')$ be the subset of σ' obtained by deleting from σ' all elements which intersect any E_w for $w' < w < w''$. Let $\sigma_0(w_1, w_2)$ be the intersection of all sets of the form $\sigma'(w', w'')$ for all w', w'' with $w_1 \leqslant w' < w'' \leqslant w_2$. This set will be the *smoothed contour* between w_1 and w_2 in σ'. It was shown in [3] that in the hyperspace topology of Γ, $\sigma_0(w_1, w_2)$ is an arc.

Let $T : M \to E_n$ be a continuous mapping defining a non-degenerate surface, then there exists a mapping T' which is light and FRÉCHET equivalent to T. Assume that T is light and let $C(t)$ be a contour defined by T, f, t in M with image of finite lenght in the sense of CESARI. This implies that all the sets E_w are continua of constancy and hence points in this case. Also each E_w is accessible from α and each prime end is an end. We can now prove the

Theorem: *Let $S = (T, M)$ be a* FRÉCHET *surface defined by a light mapping T on a 2-dimensional manifold M into n-dinensional Euclidean space E_n. Let $f : [S] \to$ Reals be continuous. Let $[\gamma]_f$ be the set of all components of contours corresponding to f in Q and whose images are of finite length.*

Then all components, with the exception of a countable number of components, will either be a point, a simple arc, or a simple closed curve.

Proof:

Let $C(t) \subset M$ be a contour satisfying the hypothesis.

Let α be a component of $D^-(t)$.

Let γ be a component of $(\alpha^* - \alpha)$.

Assume that $p \in \gamma$ is the endpoint of two distinct ends η_1, η_2 where $\eta_1 \neq \eta_2$ from $A(\alpha, t)$ to γ. Let b_1, b_2 be defining arcs for η_1, η_2 respectively such that

$b_1 - \{p\} \in A(\alpha, \gamma)$; $b_2 - \{p\} \in A(\alpha, \gamma)$ and $[b_1 - p] \cap [b_2 - p] = p'$, where p', is a point of $A(\alpha, \gamma)$. Then $b_1 \cup b_2$ is a simple closed curve in M which lies in $A(\alpha, \gamma)$ and $\beta \cap \gamma = \{p\}$. There exist ends between η_1 and η_2, in since $\eta_1 \neq \eta_2$, and there are points of γ interior to β. Suppose $\eta_1 < \eta_2$. Several cases arise.

Case I (See Figure I): There exist ends η_1', η_2' with $\eta_1' < \eta_1 < \eta_2 < \eta_2'$ ending on γ with $\omega_{\eta_1'} \neq \omega_{\eta_2'}$, $\omega_{\eta_2'} \neq p$, $\omega_{\eta_2} \neq p$ such that defining arcs exist for η_1', η_2' which together form a crosscut C for $A(\alpha, \gamma)$ and such that

(i) $\beta - (p)$ lies in the component of $A(\alpha, \gamma) - C$ which contains as boundary points the points $\omega_\eta \in \gamma$, $\eta_1 < \eta < \eta_2$.

(ii) There exists no end η_1'' between η_1 and η_1' for which $w_{\eta_1''} = w_{\eta_2'}$ and no end η_2'' between η_2 and η_2' with $\eta_2'' = \eta_1'$.

Let M be a 2- manifold. Consider a segment of ends and prime ends of A ending on γ as defined by R. E. FULLERTON [2].

For every point $p \in M$ there exists an open subset $N_p \subset M$ which is homeomorphic to an open disc in the plane whose center is the counterimage of p.

If p is a boundary point of M, then N_p is homeomorphic to an open half disc in the plane plus is bounding diameter with p the image of the mid-point of the diameter. Neighborhoods of the above type, in accordance with R. E. FULLERTON, will be called *coordinate neighborhoods* since, it is possible to introduce local co-ordinate systems at each point of M.

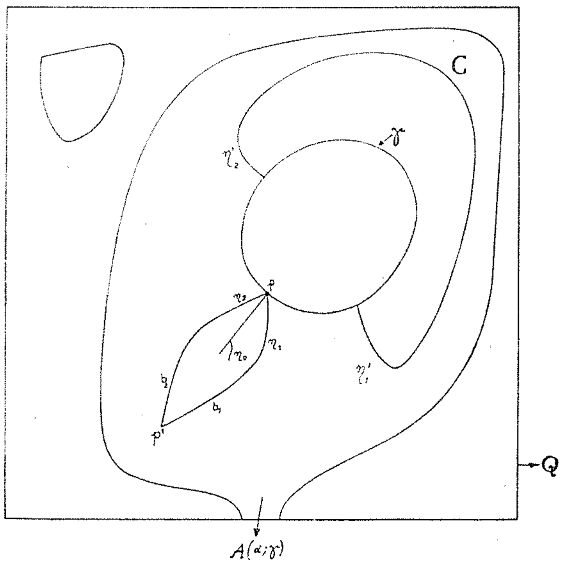


Fig. I.

Since M is compact, M can be covered by a finite number of these neighborhoods. Let U be one of these connected neighborhoods such that $p \in U \subset D^{-(t)}$ with the two ends η_1 and η_2 ending on γ at p. We can order the ends on γ in U by using a segment of ends, FULLERTON [2]. Since $\eta_1 < \eta_2$ there exists an end η_0 with $\eta_1 < \eta_0 < \eta_2$.

Following R. E. FULLERTON's proof it can be shown that all contours but a countable number of them are locally smooth i.e. by using the first smoothing method described by R. E. FULLERTON [2] the smoothed portion of γ between η_1 and η_0 is an arc in the hyperspace topology. Since the hyperspace topology coincides with the ordinary topology in Q the smoothed portion of γ yields also an arc in Q. Smoothing γ between η_1' and η_1 and between η_2' and η_2 yields altogether three arcs with not more but one point in common which is the endpoint of each arc. This configuration was called « triod » by R. L. MOORE and by one of his theorems, there exist at most countably many distinct triods in the plane.

Case I′ (See Figure I′): is the same as Case I except that the order of the ends is reversed, i.e. $\eta_1 < \eta_1' < \eta_2' < \eta_2$.

The same arguments can be applied in this case after modifiying the order relations accordingly.

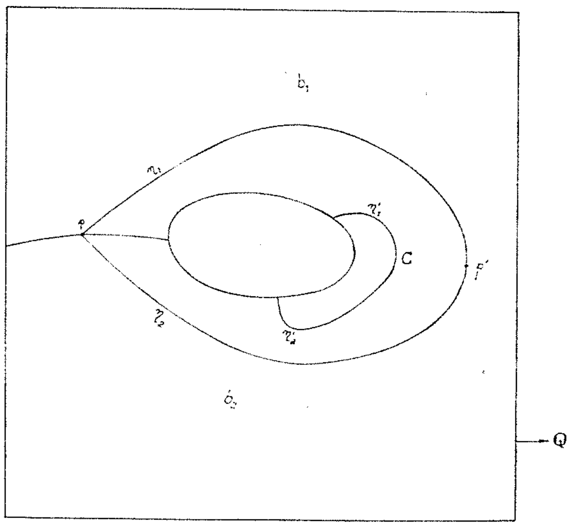


Fig. I′

Case II (See Figure II): Let η_1 be the first end on γ. There exists an end $\eta_2' < \eta_2$ with $w_{\eta_2'} \neq p$ and a cross cut C from Q^* to $w_{\eta_2'}$ which contains a defining arc for η_2' such that $\beta-(p)$ lies in the component of $A(\alpha, \gamma)-C$ which includes among its boundary points the points $w_\eta \in \gamma$ for which $\eta_1 < \eta < \eta_2$.

For manifolds without boundary there exists no Case II. Let M be a manifold with rectifiable boundary.

Let M be covered by coordinate neighborhoods. Since M is compact M can be covered by a finite number of neighborhoods.

Let η_0 be an end such that $\eta_1 < \eta_0 < \eta_2$. Smoothing between the ends η_0 and η_2 and between η_2 and η_2' on γ yields as in Case I two arcs τ_1, τ_2. The arc τ_1 has initial point w_{η_0}. The arc τ_2 has initial point $w_{\eta_2'} \neq w_{\eta_0}$ and each arc has terminal point p.

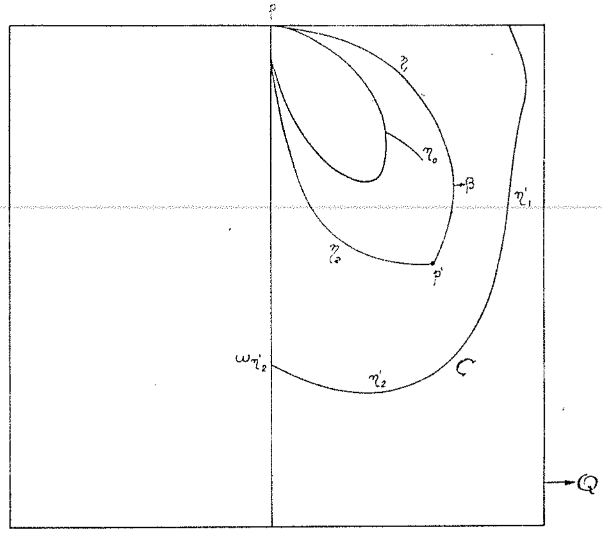
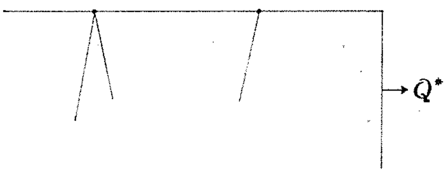


Fig. II.

If $\tau_1 \cap \tau_2$ contains a point other than p, then $\tau_1 \cap \tau_2$ contains a triod and a set of these components which contains triods is at most countable.

If $\tau_1 \cap \tau_2 = p$ the arcs intersect only on Q^*. A configuration consisting of two arcs with this property will be called in accordance with R. E. FULLERTON a « V set ». Since all contour components are distinct, no two such sets can have points in common. To show now that there are only countably many V sets in Q.



Assume that $V \cap Q^*$ contains points other than p. There two possibilities. Either V contains a V-set V' such that $V' \cap Q^*$ is a single point or a subarc of V lies in Q^*. The boundary of Q contains only countably many of these distinct subarcs and therefore only countably two possibilities many contour components can contain V-sets of this type. Each of the remaining V-sets can be assumed to intersect the boundary of Q in but one point. We will show that there exists only a countable number of this type.

Let g be a point of Q^*. Let N_g be an open half-disc neighborhood of g and let d_g be the bounding diameter with g interior to d_g. If l is the boundary of M then l can be covered by a finite number of half-disc coordinate neighborhoods N. Let d be the union of the bounding diameters of N. Parallel to d consider a sequence of converging line segments $\{d_n\}$ that converge to d.

Then if for the U set V, V and Q^* have a single point in common which lies on d, one of the segments d_n must intersect both arcs τ_1 and τ_2 of V. The union $\tau_1 \cup \tau_2 \cup d_n$ will bound an open set $G(V) \subset Q$. Also, since all the V sets are distinct if $V_1 \neq V_2$ then $G(V_1) \cap G(V_2) = 0$. Since there can be only countably many disjoint open sets in the plane, only countably many V sets in disc D can have the vertex point on d.

Case III (See Figure III): As in Case I there exist two distinct ends $\eta_1' \neq \eta_2'$ with $\eta_1' < \eta_1 < \eta_2 < \eta_2'$ satisfying that for every choice of η_1', η_2' there exists an end η_1'', $\eta_1' < \eta_1'' < \eta_1$, η_1'', η_2' satisfying that $\beta - p$ lies in the component of

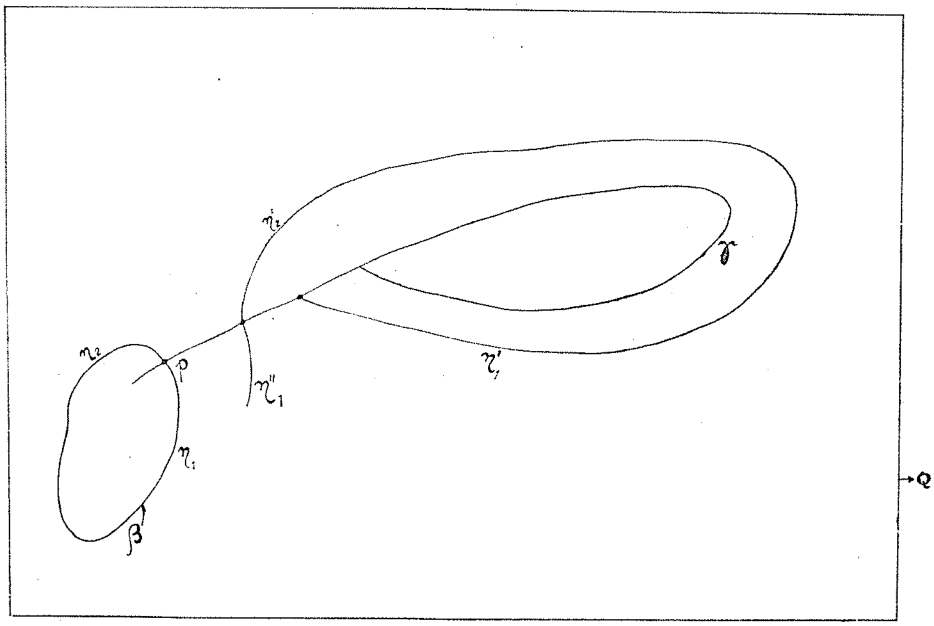


Fig. III.

$A(\alpha, \gamma) - C$ which contains as boundary points the points $w_\eta \in \gamma$, where $\eta_1 < \eta < \eta_2$ but $w_{\eta_1''} = w_{\eta_2'}$ or an end η_2'', $\eta_2 < \eta_2'' < \eta_2'$; η_1', η_2'' satisfying (i) and $w_{\eta_2''} = w_{\eta_1'}$.

First we will consider a 2-manifold M *without boundary*. For example we could think of M as a torus and Q as a band which goes around the torus.

Assume first that each point of γ is the endpoint of exactly one end. Consider a segment of ends and prime ends of A ending on γ as defined by R. E. FULLERTON.

Since M is a two dimensional manifold then for every point p of M there exists an open subset $N_p \subset M$ which is homeomorphic to an open disc in the plane whose center is the counterimage of P. (The N_p are called coordinate neighborhoods.) Also M is compact hence M can be covered by a finite number of these neighborhoods.

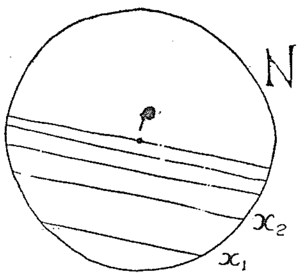
Let γ be covered by a set of coordinate neighborhoods in M. Let N be one of these neighborhoods. Then $(N - \gamma) \cap A$ will consist of components (possibly infinitely many) each having points of γ on its boundary.

It will now be shown that γ cannot « terminate » in any neighborhood. The following cases may occur:

(1) γ is contained entirely in one neighborhood N. Then this reduces to the planar case which has already been discussed by R. E. FULLERTON. Thus in the following section we consider only the significant case in which $\gamma \notin N$.

(2) Assume now that γ has been chosen and that γ does not lie completely in any coordinate neighborhood. For each point $p \in \gamma$ let N_p be a coordinate neighborhood containing p. Since γ is a closed subset of M, is compact and there exists a finite family $\{N^K\}$ $(K = 1, 2, ...)$, chosen from the family $\{N_p\}$ which covers γ.

We will show that there is no end η with endpoint p on γ such that η is the last end on the segment of γ containing p. Now we can select a neighborhood N_p, $N_p \subset N$ such that p is interior to N_p and N_p does not contain any other component of γ but the one which contains p.



Also p is not a limit point of a sequence of disjoint components of $N \cap \gamma$. Since otherwise let $p \in N^0$ and C_i $(i = 1, 2, ...)$ denote the disjoint components of $N \cap \gamma$. Then $C_i \cap N^* \neq 0$ for all i. Let $x_i \in (C_i \cap N)$, all i.

Then $\{X_i\}$ is an infinite bounded sequence which lies in the closed set γ and hence by the BOLZANO-WEIERSTRASS theorem a subsequence of $\{X_i\}$ converges to X'. By an argument involving the ZORETTI theorem, it can be shown that X' lies in the same component C_p which contains p.

Also $x' \neq p$ since all points x_i $(i = 1, 2, ...)$ are boundary points of N implies that x' is a boundary point. But p is interior to N. Hence $x' \neq p$ and C_p represents a non-degenerate limiting continuum.

According to CESARI [2] the limiting continuum C_p forms a continuum of constancy.

This yields the desired contradiction since by hypothesis T is a light function and all continua of constancy must then be points. Hence if p is interior to N then p is not a limit point of disjoint components of $\gamma \cap N$.

A similar argument can beused when p is a boundary point of N by selecting another neighborhood N'_p with p interior to N'_p.

Since p is not a limit point of disjoint components of $N \cap \gamma$, we can find a neighborhood N_p with $p \in N_p$ which does not include any other component of $\gamma \cap N$.

Let β be the arc determining η in N_p. Now η is an admissible end and every neighborhood of β must contain points of A. Therefore $N_p \cap \beta \cap A \neq 0$.

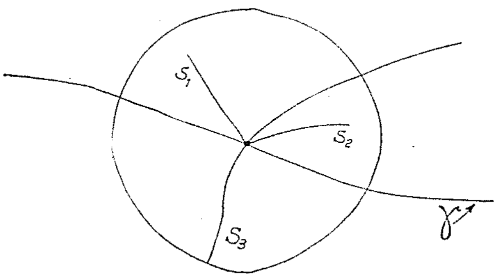
Also p is a boundary point of A. Consider a disc D with boundary C around p in N_p. Let K be a component of $D^0 \cap A$ with p as boundary point. The ordering of ends in K can be chosen to be compatible with the ordering of ends in N_p.

It has been proven by ROBERT E. FULLERTON [1] that K is a simple connected region.

Now p is interior to D and hence interior to some segment of ends of K. But if we assume that η is the last end, then all ends following η in K have to end on C.

But C is bounded away from p, since p is interior to C, which yields the desired contradiction.

We can show that at most two ends end at p. For assume otherwise i.e. that there are three ends η_1, η_2, and η_3 ending on p with $p = w_{\eta_1} = w_{\eta_2} = w_{\eta_3}$. We can choose the three ends such that $\eta_1 < \eta_2 < \eta_3$. Then let S_1, S_2, and S_3 be the arcs defining the end η_1, η_2, and η_3 respectively. Again let N_p be a coordinate neighborhood of p. Ends of A ending at p can be defined in $N_p \cap A$.

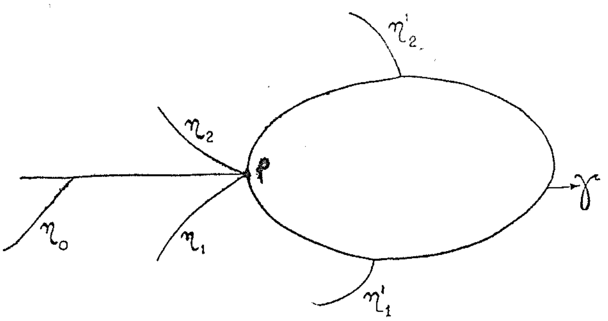


We can choose N_p such that the three arc are entirely in the interior of N_p. N_p can be chosen such that part of the boundary will have to lie between each two arcs since otherwise the two arcs would be equivalent i.e. for every N_p there would exist a subarc S' with $S_1 S' = (w)$, $S_2 S' = (w')$ and the open JORDAN region whose boundary is $S_1 + S' + S_2$ is contained in N_p. Therefore the simple

JORDAN region which is bounded by S_1, S_2 and a portion of boundary of N_p can be disconnected by γ. The same statement applies to the region bounded by S_2, S_3 and a portion of N_p as well as to the region which is bounded by S_1, S_3 and N_p. Since S_1, S_2, and S_3 are three arcs which have only one point in common, namely p_1 this configuration contains a triod and according to R. L. MOORE [5] there are but a countable number of them. Ergo the case where more than two ends end on p can be discarded in our considerations.

We know now that there is at least one point p on γ which is the endpoint of two ends. To show further that every point of γ, with two possible exceptions, is the endpoint of two ends. Let η_1, η_2 be the two ends with $w_{\eta_1} = w_{\eta_2} = p$ with $\eta_1 < \eta_2$. Let N_p a coordinate neighborhood of p. Ends of A ending on p to be defined as previously. This can be done for any point p of γ. Since γ is a closed subset of M, γ is compact and there exists a finite family $\{N_g\}$ $(g = 1, 2, \ldots, n)$ which is contained in the collection $\{N_p\}$ covering γ. Assume that there are two ends η_1', η_2' in N_p with $\eta_1 < \eta_1' < \eta_2' < \eta_2$ for which there exist no ends η_1'', η_2'' with $w_{\eta_1''} = w_{\eta_1'}$ and $w_{\eta_2''} = w_{\eta_2'}$. (See figure.) Let η_0 be an end with $\eta_1 < \eta_0 < \eta_2$. Since $\eta_1 < \eta_2$ and the order is cyclic there exists such an end.

As it was done in Case I, smoothing og γ between η_1' and η_1; η_2' and η_2; η_0 and η_1 yields 3 arcs with but one point in common. As before this gives rise to a triod and such cases can occur at most countably many times. After deleting these cases the only other case that one has left to consider is the one where there exists at most one end η' between η_1 and η_2 where $\eta_1 < \eta_1' < \eta_2$ for which there is no end η_0 with $w_{\eta_0} = w_{\eta'}$. Since a corresponding situation exists for the interval $\eta_2 < \eta < \eta_1$ one then can assume that all ends occur in distinct pairs with two possible exceptins. Now consider the following.



Assume that in the interval $\eta_1 < \eta < \eta_2$ the ends occur in distinct pairs with one exception. We will say that two ends belong to the same pair if they have the same endpoint on γ.

Let A be the collection of all ends on « one side of γ » and let A' be the set of all ends on « the other side of γ »; i.e. if η and η' are a pair of ends with

$w_\eta = w_{\eta'}$ on γ and if $\eta < \eta'$ then let η be in A and let η' be in A'. Similarly for ends in $\eta_2 \leqslant \eta \leqslant \eta_1$.

Again let γ be covered by a finite number of co-ordinate neighborhoods. Consider the class A and A' for the interval $\eta_1 \leqslant \eta \leqslant \eta_2$. Every end of the interval with one exception is contained in A or A' and every end of A is less than every end of A'. Therefore A and A' define a prime end, let us denote this end by η_0. Also η_0 does not belong to any pair since if η_0, η_0' were a pair with $\eta_0 < \eta_0'$ then there would be an end η^* between η_0 and η_0'. But this yeilds a contradiction since then either η^* is in A but $\eta^* > \eta_1$ or otherwise $\eta^* \in A'$ and $\eta^* < \eta_0'$ which contradicts the fact that η_0 is defined by A and A'. Therefore η_0 does not belong to any pair of ends in the interval $\eta_1 \leqslant \eta_0 \leqslant \eta_2$. Also in the interval $\eta_2 \leqslant \eta \leqslant \eta_1$ there exists a corresponding end $\bar{\eta}_0$ which is unique. Since these two intervals $\eta_1 \leqslant \eta \leqslant \eta_2$ and $\eta_2 \leqslant \eta \leqslant \eta_1$ include all ends ending on points of γ. Consider the endpoints of ends on γ in the interval $[\bar{\eta}_0, \eta_0]$. These include all the points of γ and each point of γ is the endpoint of exactly one end terminating on γ. Therefore portion of γ between η_0 and $\bar{\eta}_0$ is smooth and is an arc.

Hence it has been shown in Case III that γ is an arc or contains a triod ora a V set. Omitting all components γ of the later variety, all but a countable family of the components are arcs.

In all other cases but those considered, the contour γ is smooth between any two of its points. For any contour γ with first and last ends terminating on it, the V set is an arc. Given any two points of γ, γ consists of 2 arcs with but 2 points in common and is therefore a simple closed curve in M. Hence all contours γ except possibly a countable number are arcs, points or simple closed curves.

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