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A Problem in Partitions Related to the Stirling Numbers. (**)

1. - Introduction.

Let

$$S(n,\ r) = \frac{1}{r!}\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}s^n \tag{1.1}$$

denote the STIRLING number of the second kind. In a recent paper [1] the writer has discussed some properties of the polynomial

$$A_n(x) = \sum_{r=0}^{n} S(n,\ r)\, x^r\ ; \tag{1.2}$$

in particular the factorization (mod 2) of $A_n(x)$ is determined. If we put

$$c_{nr} = S(n+1,\ r+1), \tag{1.3}$$

it is proved that

$$\begin{cases} c_{n,2r} \equiv \binom{n-r}{r} \pmod 2 & (0 \leqslant 2r \leqslant n) \\ c_{n,2r+1} \equiv \binom{n-r-1}{r} \pmod 2 & (2r+1 \leqslant n)\,. \end{cases} \tag{1.4}$$

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Also if $\theta_0(n)$ denotes the number of odd $c_{n,2r}$ and $\theta_1(n)$ denotes the number of odd $c_{n,2r+1}$, then by (1.3) and (1.4)

$$\theta_0(2n+1) = \theta_0(n), \qquad \theta_0(2n) = \theta_0(n) + \theta_0(n-1) \tag{1.5}$$

and

$$\theta_1(n+1) = \theta_0(n) . \tag{1.6}$$

Moreover it follows easily from (1.5) that

$$\sum_{n=0}^{\infty} \theta_0(n)\, x^n = \prod_{n=0}^{\infty} (1 + x^{2^n} + x^{2^{n+1}}). \tag{1.7}$$

The first few values of $\theta_0(n)$ are easily computed.

$\theta_0(0) = 1,\ \theta_0(1) = 1,\ \theta_0(2) = 2,\ \theta_0(3) = 1,\ \theta_0(4) = 3,\ \theta_0(5) = 2,$

$\theta_0(6) = 3,\ \theta_0(7) = 1,\ \theta_0(8) = 4,\ \theta_0(9) = 3,\ \theta_0(10) = 5,\ \theta_0(11) = 2,$

$\theta_0(12) = 5,\ \theta_0(13) = 3,\ \theta_0(14) = 5,\ \theta_0(15) = 1 .$

Conversely if we put

$$G(x) = \prod_{n=0}^{\infty} (1 + x^{2^n} + x^{2^{n+1}}) \tag{1.8}$$

then

$$G(x) = (1 + x + x^2)\, G(x^2)$$

and (1.5) follows at once.

From (1.7) it is clear that $\theta_0(n)$ can also be defined as the number of partitions

$$n = n_0 + n_1 \cdot 2 + n_2 \cdot 2^2 + \dots \qquad (0 \leqslant n_j \leqslant 2) \tag{1.9}$$

subject to the following conditions:

(i) if $n_0 = 1$ then $n_1 \leqslant 1$,

(ii) if $n_1 = 2$ then $n_2 \leqslant 1$,

(iii) if $n_2 = 2$ then $n_3 \leqslant 1$,

and so on.

The object of the present paper is to obtain some additional properties of $\theta_0(n)$. A summary of the results will be found at the end of the paper.

2. - We observe first that (1.4) can be proved rapidly by making use of the familiar formula (see for example [**3**, p. 45, ex. 15])

$$\sum_{n=0}^{\infty} S(n+r,\ r)\, x^n = \frac{1}{(1-x)(1-2x)\ldots(1-rx)} .$$

Using (1.3) this becomes

$$\sum_{n=0}^{\infty} c_{n+r,r}\, x^n = \frac{1}{(1-x)(1-2x)\ldots(1-(r+1)x)} . \tag{2.1}$$

If

$$A(x) = \sum_{n=0}^{\infty} a_n\, x^n, \qquad B(x) = \sum_{n=0}^{\infty} b_n\, x^n$$

are two (formal) power series with integral coefficients, the statement

$$A(x) \equiv B(x) \pmod{m}$$

means

$$a_n \equiv b_n \pmod{m} \qquad (n = 0,\ 1,\ 2,\ \ldots),$$

Thus if we replace r by $2r$, (2.1) implies

$$\sum_{n=0}^{\infty} c_{n+2r,2r}\, x^n \equiv (1-x)^{-r-1} \pmod{2} .$$

Since

$$(1-x)^{-r-1} = \sum_{n=0}^{\infty} \binom{n+r}{r} x^n,$$

we get

$$c_{n+2r,2r} \equiv \binom{n+r}{r} \pmod{2},$$

wich is equivalent to the first of (1.4). Similarly (2.1) yields

$$\sum_{n=0}^{\infty} c_{n+2r+1,2r+1}\, x^n \equiv (1-x)^{-r-1} \pmod{2},$$

so that

$$c_{n+2r+1,2r+1} \equiv \binom{n+r}{r} \pmod 2,$$

which is equivalent to the second half of (1.4).

We turn next to the function $\theta_0(n)$. If $r \geqslant 1$, $m \geqslant 1$ it follows from the first of (1.5) that

$$\theta_0(2^r m - 1) = \theta_0(2^{r-1} m - 1) .$$

This evidently implies

(2.2) $$\theta_0(2^r m - 1) = \theta_0(m - 1) \qquad (r \geqslant 1, \ m \geqslant 1) .$$

In particular, when $m = 1$, we get

(2.3) $$\theta_0(2^r - 1) = 1 .$$

Similarly from the second of (1.5) we get

$$\theta_0(2^r m) = \theta_0(2^{r-1} m) + \theta_0(2^r m - 1),$$

which by (2.2) reduces to

(2.4) $$\theta_0(2^r m) = \theta_0(2^{r-1} m) + \theta_0(m - 1) \qquad (r \geqslant 1, \ m \geqslant 1) .$$

Repeated application of (2.4) yields

(2.5) $$\theta_0(2^r m) = \theta_0(m) + r\, \theta_0(m - 1) \qquad (r \geqslant 0, \ m \geqslant 1) .$$

In particular, for $m = 1$, (2.5) reduces to

(2.6) $$\theta_0(2^r) = r + 1 .$$

We note also that by the first of (1.5)

(2.7) $$\theta_0(2^r m + 1) = \theta_0(2^{r-1} m) \qquad (r \geqslant 1, \ m \geqslant 1) ,$$

and in particular, by (2.6),

(2.8) $$\theta_0(2^r + 1) = r \qquad (r \geqslant 1) .$$

In (1.4) take $m = 2^s + 1$, $s > 1$, so that

$$\theta_0(2^{r+s} + 2^r) = \theta_0(2^s + 1) + r\,\theta_0(2^s)\,.$$

Making use of (2.5) and (2.8), this becomes

$$\theta_0(2^{r+s} + 2^r) = rs + r + s \qquad (r \geqslant 0,\ s \geqslant 1)\,. \tag{2.9}$$

Similarly we have for $s \geqslant 1$

$$\theta_0(2^{r+s} - 2^r) = \theta_0(2^s - 1) + r\,\theta_0(2^s - 2).$$

Since by (1.5)

$$\theta_0(2^s - 2) = \theta_0(2^{s-1} - 1) + \theta_0(2^{s-1} - 2),$$

it follows that

$$\theta_0(2^s - 2) = s \qquad (s \geqslant 1)\,, \tag{2.10}$$

so that

$$\theta_0(2^{r+s} - 2^r) = 1 + r\,s \qquad (r \geqslant 0,\ s \geqslant 1)\,. \tag{2.11}$$

Note that (2.11) includes both (2.3) and (2.5).

3. - Returning to (2.5) we take

$$m = 2^{s+t} + 2^s + 1 \qquad (s \geqslant 1,\ t \geqslant 1)\,.$$

We get

$$\theta_0(2^{r+s+t} + 2^{r+s} + 2^r) = \theta_0(2^{r+t} + 2^s + 1) + r\,\theta_0(2^{s+t} + 2^s)\,.$$

Since for $s \geqslant 1$

$$\theta_0(2^{s+t} + 2^s + 1) = \theta_0(2^{s+t-1} + 2^{s-1})\,,$$

it follows from (2.9) that

$$\theta_0(2^{r+s+t} + 2^{r+s} + 2^r) = s + t - 1 + (s-1)t + r(s + t + s\,t)\,,$$

so that

$$\theta_0(2^{r+s+t} + 2^{r+s} + 2^r) = (r+1)s + r\,t + (r+1)s\,t - 1 \qquad (r \geqslant 0,\ s \geqslant 1,\ t \geqslant 1)\,. \tag{3.1}$$

In order to get a general result of this kind we put

$$n = 2^{r_0} + 2^{r_0+r_1} + \ldots + 2^{r\ +r_1 \cdots +r_k}\,, \tag{3.2}$$

where

$$r_0 \geqslant 0,\ r_1 \geqslant 1,\ \ldots,\ r_k \geqslant 1\,.$$

It is also convenient to put

$$n_j = 2^{r_j} + 2^{r_j+r_{j+1}} + \ldots + 2^{r_j+\ \cdots\ +r_k} \qquad (0 \leqslant j \leqslant k), \tag{3.3}$$

so that $n_0 = n$. It is clear from (3.3) that

$$n_j = 2^{r_j}\,(1 + n_{j+1}) \qquad (0 \leqslant j < k)\,. \tag{3.4}$$

Thus (2.5) becomes

$$\theta_0(n) = \theta_0(1 + n_1) + r_0\,\theta_0(n_1)\,. \tag{3.5}$$

Since n_1 is even we have

$$\theta_0(1 + n_1) = \theta_0(n/2)\,.$$

But

$$\theta_0(n_1) = \theta_0\left(\frac{n_1}{2}\right) + \theta_0\left(\frac{n_1}{2} - 1\right)\,,$$

so that (3.5) becomes

$$\theta_0(n) = (1 + r_0)\,\theta_0(n_1) - \theta_0\left(\frac{n_1}{2} - 1\right).$$

Now by (2.2) and (3.4)

$$\theta_0\left(\frac{n_1}{2} - 1\right) = \theta_0\left(2^{r_1-1}\,(1 + n_2) - 1\right) = \theta_0(n_2).$$

Therefore finally

$$\theta_0(n) = (1 + r_0)\ \theta_0(n_1) - \theta_0(n_2)\ . \tag{3.6}$$

This formula was stated without proof in [**1**].

4. - With the notation (3.4) we may think of (3.6) as a recurrence formula. In general we have

$$\theta_0(n_j) = (1 + r_j)\ \theta_0(n_{j+1}) - \theta_0(n_{j+2}) \qquad (0 \leqslant j \leqslant k). \tag{4.1}$$

A second formula of a similar kind can be stated. Put

$$m_j = 2^{r_0} + 2^{r_0+r_1} + \ldots + 2^{r_0+r_1+\ldots+r_j}\ , \tag{4.2}$$

where as before

$$r_0 \geqslant 0,\ r_1 \geqslant 1,\ \ldots,\ r_j \geqslant 1\ .$$

Then we have

$$\theta_0(m_j) = (1 + r_j)\ \theta_0(m_{j-1}) - \theta_0(m_j - 2) \qquad (j \geqslant 1)\ , \tag{4.3}$$

where $m_{-1} = 0$. Indeed (4.3) is equivalent to (4.1) as follows by comparison of (4.2) and (3.3).

A more general relation is

$$\theta_0(n) = \theta_0(m_j)\ \theta_0(n_{j+1}) - \theta_0(m_{j-1})\ \theta_0(n_{j+2}) \qquad (0 \leqslant j < k) \tag{4.4}$$

where $m_{-1} = n_{k+1} = 0$. This is easily proved by induction with respect to j.

The recurrence (4.3) suggests a connection with continued fractions. However it is more convenient to make use of *continuants* (see [**2**, pp. 466-474]). We recall the following definition.

If $a_0, a_1, \ldots, a_k, b_1, \ldots, b_k$ are indeterminates and p_k is defined by means of the set of equations

$$p_j = a_j\, p_{j-1} + b_j p_{j-2} \qquad (1 \leqslant j \leqslant k) \tag{4.5}$$

together with the initial conditions $p_{-1} = 1$, $p_0 = a_0$, then we put

$$p_k = K\begin{pmatrix} & b_1, & \ldots, & b_k \\ a_0, & a_1 & \ldots, & a_k \end{pmatrix}. \tag{4.6}$$

Comparing (4.5) with (4.3) we have

$$a_j = 1 + r_j, \qquad b_j = -1 \qquad (1 \leqslant j \leqslant k)\,,$$

so that

$$\theta_0(n) = K\begin{pmatrix} -1, \dots, -1 \\ 1 + r_0,\ 1 + r_1,\ \dots,\ 1 + r_k \end{pmatrix};$$

for brevity we write this in the form

$$\theta_0(n) = K'(1 + r_0,\ 1 + r_1,\ \dots,\ 1 + r_k)\,. \tag{4.7}$$

By known properties of continuants we have for example

$$K'(1 + r_0,\ 1 + r_1,\ \dots,\ 1 + r_k) = K'(1 + r_k,\ 1 + r_{k-1},\ \dots,\ 1 + r_0)\,. \tag{4.8}$$

We may also mention the determinantal representation

$$\theta_0(n) = \begin{vmatrix} 1 + r_0 & -1 & 0 & \cdot & \cdot & \cdot & \cdot \\ -1 & 1 + r_1 & -1 & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & 1 + r_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 + r_k \end{vmatrix}. \tag{4.9}$$

5. - When

$$r_0 = r_1 = \dots = r_1 = r \tag{5.1}$$

we can find a simple explicit formula for $\theta_0(n)$. Indeed in this case (4.3) becomes

$$\theta_0(m_j) = (1 + r)\,\theta_0(m_{j-1}) - \theta_0(m_{j-2})\,, \tag{5.2}$$

a recurrence with constant coefficients and characteristic equation

$$x^2 - (1 + r)\,x + 1 = 0\,.$$

Therefore if we put

$$\varepsilon = \frac{1 + r + \sqrt{(1 + r)^2 - 4}}{2}$$

and make use of the initial conditions

$$\theta_0(m_{-1}) = 1, \qquad \theta_0(m_0) = r + 1,$$

we find easily that

$$\theta_0(n) = \frac{\varepsilon^{k+2} - \varepsilon^{-k-2}}{\varepsilon - \varepsilon^{-1}} \qquad (r \neq 1) , \tag{5.3}$$

where

$$n = 2^r \frac{2^{(k+1)r} - 1}{2^r - 1} .$$

This formula holds for all $r \geqslant 2$. For $r = 1$, however, the recurrence (5.2) with the initial conditions

$$\theta_0(m_{-1}) = 1, \qquad \theta_0(m_0) = 2$$

has the solution

$$\theta_0(n) = k + 2 \qquad (r = 1) . \tag{5.4}$$

This is equivalent to

$$\theta_0\big(2(2^k - 1)\big) = k + 1 ,$$

which is in agreement with (2.11).

If in place of (5.1) we assume that

$$r_0 \geqslant 1, \qquad r_1 = \ldots = r_k = r, \tag{5.5}$$

then by (3.6)

$$\theta_0(n) = (1 + r_0)\, \theta_0(n_1) - \theta_0(n_2) ,$$

where now

$$n_1 = 2^r \frac{2^{kr} - 1}{2^r - 1}, \qquad n_2 = 2^r \frac{2^{(k-1)r} - 1}{2^r - 1} .$$

Thus by (5.3) and (5.4) we get

$$\theta_0(n) = (1 + r_0) \frac{\varepsilon^{k+1} - \varepsilon^{-k-1}}{\varepsilon - \varepsilon^{-1}} - \frac{\varepsilon^k - \varepsilon^{-k}}{\varepsilon - \varepsilon^{-1}} \qquad (r \neq 1), \tag{5.6}$$

$$\theta_0(n) = r_0 (k + 1) + 1 \qquad (r = 1) , \tag{5.7}$$

where in both (5.6) and (5.7)

$$n = 2^{r_0} \frac{2^{(k+1)r} - 1}{2^r - 1} .$$

6. – It is evident from (1.7) that

$$\theta_0(n) \geqslant 1 ; \tag{6.1}$$

in view of (2.3) this lower bound cannot be improved. It is of some interest to find all n such that

$$\theta_0(n) = 1 . \tag{6.2}$$

If $n = 2m > 0$ we have

$$\theta_0(2m) = \theta_0(m) + \theta_0(m-1) \geqslant 2 .$$

Thus the solutions of (6.2) are necessarily odd. If $n = 2m + 1$, (6.2) implies $\theta_0(m) = 1$, so that m is also odd. Proceeding in this way we conclude that

$$n = 2^r - 1 . \tag{6.3}$$

We see therefore that (6.2) is satisfied if and only if n is of the form (6.3) Another way of stating this result is

$$\theta_0(n) \geqslant 2 \qquad (n \neq 2^r - 1) . \tag{6.4}$$

If

$$\theta_0(n) = 2 \tag{6.5}$$

and $n = 2m$, we get

$$\theta_0(m) + \theta_0(m-1) = 2,$$

so that $\theta_0(m) = \theta_0(m-1) = 1$. Since either m or $m-1$ is even we must have $m = 1$. Next if $n = 2m + 1$ we get $\theta_0(m) = 2$; if m is even it follows as before that $m = 2$. Continuing in this way we see that the solutions of (6.5) are given by

$$n = 2, 5, 11, 23, 47, \ldots .$$

If n_k is the r-th term in this sequence, we have

$$n_{r+1} = 2n_r + 1 \qquad (n_1 = 2);$$

it follows that

(6.6) $$n_r = 2^r + 2^{r-1} - 1 .$$

Thus (6.5) is satisfied if and only if n is of the form (6.6). Note that n_r is odd for $r > 1$.

Finally if

(6.7) $$\theta_0(n) = 3$$

and $n = 2m$ we get

$$\theta_0(m) + \theta_0(m-1) = 3 .$$

If

(6.8) $$\theta_0(m) = 2, \qquad \theta_0(m-1) = 1,$$

then by the results just proved

$$m - 1 = 2^r - 1, \qquad m = 2^s + 2^{s-1} - 1;$$

thus

$$2^s + 2^{s-1} - 1 = 2^r,$$

so that $r = s = 1$, $n = 4$. If

(6.9) $$\theta_0(m) = 1, \qquad \theta_0(m-1) = 2,$$

we get

$$m = 2^r - 1, \qquad m - 1 = 2^s + 2^{s-1} - 1,$$

so that

$$2^s + 2^{s-1} = 2^r - 1 .$$

This implies $s = 1$, $r = 2$, $m = 6$. Thus the only even solutions of (6.7) are $n = 4,\ 6$.

Next if $n = 2m + 1$, (6.7) implies $\theta_0(m) = 3$. Comparing this with the discussion of (6.5) we see that the solutions of (6.7) are given by the following two sequences:

$$n = 4,\ 9,\ 19,\ 39,\ \dots;\ 6,\ 13,\ 17,\ 55, \dots .$$

As before we have the recurrence

$$n_{r+1} = 2n_r + 1 .$$

We find that

$$n = 2^{r+1} + 2^{r-1} - 1 \qquad (r = 1,\ 2,\ 3,\ \dots)\,, \tag{6.10}$$

$$n = 3.2^{r} + 2^{r-1} - 1 \qquad (r = 1,\ 2,\ 3,\ \dots)\,. \tag{6.11}$$

Thus (6.7) is satisfied if and only if n is of the form (6.10) or (6.11). The sequences evidently have no terms in common.

The equation

$$\theta_0(n) = t\,, \tag{6.12}$$

where t is an assigned positive integer, is always solvable. For example by (2.6) one solution is $n = 2^{t-1}$. From the above discussion it is clear that for $t > 1$ there exist even solutions

$$n = e_1,\ e_2,\ e_3,\ \dots \tag{6.13}$$

and that all solutions are given by

$$e_j,\quad 2e_j + 1,\quad 4e_j + 3,\quad 8e_j + 7,\ \dots \qquad (j = 1,\ 2,\ 3, \dots). \tag{6.14}$$

Moreover we can assert that the number of even solutions (6.13) is finite. This is an immediate consequence of the following inequality

$$\theta_0(2n) \geqslant r + 1 \qquad (2^{r-1} \leqslant n < 2^{r}). \tag{6.15}$$

We shall prove (6.15) by induction with respect to r. For $r = 1$, $n = 1$ the formula is obvious. Assume that (6.15) holds up to and including the value $r - 1$. If $n = 2^{r-1}$ there is nothing to prove. We shall accordingly assume that $n > 2^{r-1}$. Since

$$\theta_0(2n) = \theta_0(n) + \theta_0(n-1)\,,$$

we get

$$\theta_0(2n) \geqslant 1 + \theta_0(n) \qquad (n \text{ even}),$$

$$\theta_0(2n) \geqslant 1 + \theta_0(n-1) \qquad (n \text{ odd}).$$

Since $n \geqslant 2^{r-1}$, $\frac{1}{2}(n-1) \geqslant 2^{r-2}$, so that by the inductive hypothesis

$$\theta_0(n) \geqslant r \qquad (n \text{ even}),$$

$$\theta_0(n-1) \geqslant r \qquad (n \text{ odd}).$$

This evidently completes the proof of (6.15).

It follows from (6.15) that if

$$\theta_0(2n) = t, \tag{6.16}$$

then

$$n < 2^{t-1}. \tag{6.17}$$

For example, when $t = 3$, we get $n < 4$, which agrees with the results obtained above for (6.7). Moreover (6.17) furnishes an upper bound for the number of even solutions of (6.12). Thus (6.7) is satisfied if and only if n is of the form (6.10) or (6.11). The sequences evidently have no terms in common.

7. – It is not difficult to get an upper bound for $\theta_0(n)$. With the notation (3.2), (4.7) we have, using (3.6),

$$\theta_0(n) \leqslant (1 + r_0)\, \theta_0(n_1).$$

It follows that

$$\theta_0(n) \leqslant (1 + r_0)(1 + r_1) \dots (1 + r_k) \tag{7.1}$$

and that the inequality is strict for $k > 0$.

The bound (7.1) can be replaced by a bound that depends only on n and k. We shall assume $k \geqslant 1$. Then by (3.2)

$$(1 + r_0) + (1 + r_1) + \dots + (1 + r_k) \leqslant \log_2 n,$$

so that

$$\frac{(1 + r_0) + (1 + r_1) + \dots + (1 + r_k)}{k + 1} \leqslant \frac{\log_2 n}{k + 1} .$$

Since the geometric mean of $k + 1$ positive numbers $\leqslant$ the arithmetic mean we get

$$\{ (1 + r_0)(1 + r_1) \dots (1 + r_k) \}^{1/(k+1)} \leqslant \frac{\log_2 n}{k + 1} .$$

It follows that

$$\theta_0(n) < \left(\frac{\log_2 n}{k + 1}\right)^{k+1} . \tag{7.2}$$

We remark that in the special case

$$r_0 = r_1 = \dots = r_k = r , \tag{7.3}$$

the upper bound in (7.2) cannot be improved. More precisely, when (7.3) holds and k is fixed, we have

$$\theta_0(n) \sim \left(\frac{\log_2 n}{k + 1}\right)^{k+1} \qquad (r \to \infty) . \tag{7.4}$$

Indeed by (5.3)

$$\theta_0(n) \sim \varepsilon^{k+1}.$$

Since $\varepsilon \sim r$, it follows that

$$\theta_0 \sim r^{k+1} . \tag{7.5}$$

On the other hand since

$$n = 2^r \, \frac{2^{(k+1)r} - 1}{2^r - 1} ,$$

it is evident that

$$\log_2 n \sim (k + 1) \, r ,$$

so that (7.4) and (7.5) are equivalent.

8. - Summary of results.

§ 2. Proof of (1.4). Some simple properties of $\theta_0(n)$.

§ 3. Evaluation of $\theta_0(2^{r+s+t}+2^{r+s}+2^r)$ and proof of the recurrence (3.6).

§ 4. The recurrence (4.3) and the expression of $\theta_0(n)$ as a continuant. Representation as a determinant (4.9).

§ 5. Explicit evaluation when $r_0=r_1=\ldots=r_k$ or when $r_1=\ldots=r_k$.

§ 6. Solution of the equations $\theta_0(n)=1, 2, 3$. For given $t>1$, the equation

$$\theta_0(n)=t$$

has a finite number of even solutions $e_1, \ldots, e_N$, where $N=N(t)$, such that all solutions are given by

$$e_j,\ 2e_j+1,\ 4e_j+3,\ 8e_j+7,\ \ldots \qquad (j=1, \ldots, N).$$

§ 7. Upper bound for $\theta_0(n)$. An asymptotic formula for $\theta_0(n)$ when $r_0=r_1=\ldots=r_k$.

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