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On a Generalization of Bernstein Polynomials. (**)

§ 1. - Introduction.

A recent theorem of KOROVKIN [1] gives a necessary and sufficient condition that a sequence of nonnegative linear operators (1) L_n defined in $C[a, b]$ have the property that $L_n f \to f$ uniformly for all $f \in C[a, b]$. The condition is simply that $L_n f \to f$ for the particular functions $f(x) = 1$, x, and x^2. The proof is an adaptation of BERNSTEIN's proof that the BERNSTEIN polynomials

$$(B_n f)(x) := \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \tag{1}$$

converge uniformly to f on $[0, 1]$. KOROVKIN's theorem explains why the operators B_n defined by (1) originate in the identity

$$1 = [x + (1-x)]^n = \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu}, \tag{2}$$

and indeed why all generalizations of the BERSTEIN polynomials seem to be based on some identity such as this. See for example KAC [2], MEYER-KÖNIG and ZELLER [3], SZÁSZ [4], LORENTZ [5].

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(1) An operator L is nonnegative if $f(x) \geq 0$ (for all x) implies $(Lf)(x) \geq 0$ (for all x).

The object of this Note is to draw attention to an interesting generalization of the binomial theorem due to JENSEN [6], and to show then that a generalization of the BERNSTEIN polynomials may be based upon it. JENSEN's formula is

$$(3)\qquad (x+y+n\beta)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} x(x+\nu\beta)^{\nu-1}\,[y+(n-\nu)\beta]^{n-\nu}\,.$$

The proof of (3) starts with LAGRANGE's formula

$$\frac{\Phi(z)}{1-\dfrac{z\,f'(z)}{f(z)}} = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}\,\frac{d^\nu}{dz^\nu}\,[(f(z))^\nu\,\Phi(z)]\left[\frac{z}{f(z)}\right]^\nu$$

and proceeds by setting $\Phi(z) = e^{xz}$ and $f(z) = e^{\beta z}$.

§ 2. - The operators and their convergence.

In analogy with the BERNSTEIN polynomials we set $y = 1-x$ in (3) to obtain the following extension of (2)

$$(4)\qquad 1 = (1+n\beta)^{-n} \sum_{\nu=0}^{n} \binom{n}{\nu} x\,(x+\nu\beta)^{\nu-1}\,[1-x+(n-\nu)\beta]^{n-\nu}\,.$$

The desired generalization of the BERNSTEIN polynomials is, then, in analogy with (1)

$$(5)\qquad (P_n f)(x) := (1+n\beta)^{-n} \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\binom{n}{\nu} x\,(x+\nu\beta)^{\nu-1}\,[1-x+(n-\nu)\beta]^{n-\nu}\,.$$

It is clear that the BERNSTEIN polynomials form a special case of (5) obtained by setting $\beta = 0$. The problem we propose to solve is the following: for what values of β will the operators P_n have the property that $P_n f \to f$ uniformly on $[0, 1]$ for all $f \in C[0, 1]$?

Theorem 1. *If* $0 \leqq \beta = o(n^{-1})$ *then* $P_n f \to f$ *(uniformly) for all* $f \in C[0,1]$.

The proof can be made to depend on the following lemma.

Lemma 1. *The functions*

$$S(k,\ n,\ x,\ y) := \sum_{\nu=0}^{n} \binom{n}{\nu} (x + \nu\beta)^{\nu+k-1}\ [y + (n-\nu)\beta]^{n-\nu}$$

satisfy the reduction formula

$$S(k,\ n,\ x,\ y) = x\, S(k-1,\ n,\ x,\ y) + n\beta\, S(k,\ n-1,\ x+\beta,\ y)\,.$$

The proof of the Lemma is a straightforward calculation and is therefore omitted. By repeated use of the reduction formula, noting from (3) that $x\, S(0,\ n,\ x,\ y) = (x + y + n\beta)^n$, we may show that

$$S(1,\ n,\ x,\ y) = \sum_{\nu=0}^{n} \binom{n}{\nu} \nu!\, \beta^{\nu}\, (x + y + n\beta)^{n-\nu}\,.$$

Replacing $\nu!$ in this last expression by $\int_0^\infty t^\nu e^{-t}\, dt$ and using the binomial theorem we obtain

$$S(1,\ n,\ x,\ y) = \int_0^\infty e^{-t}(x + y + n\beta + t\beta)^n\, dt\,. \tag{6}$$

In a similar manner we may reduce $S(2,\ n,\ x,\ y)$ to the following

$$S(2,\ n,\ x,\ y) = \sum_{\nu=0}^{n} (x + \nu\beta) \binom{n}{\nu} \nu!\, \beta^{\nu}\, S(1,\ n-\nu,\ x+\nu\beta,\ y)$$

and thence to

$$S(2,\ n,\ x,\ y) = \int_0^\infty e^{-t}\, dt \int_0^\infty e^{-s}\, ds[x(x + y + n\beta + t\beta + s\beta)^n + n\beta^2 s(x + y + n\beta + t\beta + s\beta)^{n-1}]\,. \tag{7}$$

These formulas will be of use presently. In order to prove Theorem 1 it is sufficient, by KOROVKIN's result, to verify that the operator P_n is nonnegative and that $P_n f \to f$ for $f(x) = 1,\ x,$ and x^2. From the definition it is clear that P_n is nonnegative when $\beta \geqq 0$. It is also clear from (4) that $P_n 1 = 1$.

Going on to $f(t) = t$ we have

$$(P_n t)(x) = (1 + n\beta)^{-n} \sum_{\nu=0}^{n} \left(\frac{\nu}{n}\right) \binom{n}{\nu} x(x + \nu\beta)^{\nu-1} [1 - x + (n-\nu)\beta]^{n-\nu} =$$

$$= (1 + n\beta)^{-n} x \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} (x + \beta + \nu\beta)^{\nu} [1 - x + (n - 1 - \nu)\beta]^{n-1-\nu} =$$

$$= (1 + n\beta)^{-n} x\, S(1,\ n-1,\ x + \beta,\ 1 - x)\,.$$

Using (6) this last expression becomes

$$x(1 + n\beta)^{-n} \int_0^\infty e^{-t}(1 + n\beta + t\beta)^{n-1}\, dt = \frac{x}{1 + n\beta} \int_0^\infty e^{-t} \left(1 + \frac{t\beta}{1 + n\beta}\right)^{n-1} dt = : A_n x\,.$$

To show that A_n tends to 1, we make the change of variable $u = t\beta/(1 + n\beta)$ to get

$$A_{n+1} = \frac{1}{\beta} \int_0^\infty e^{-t}(1 + u)^n\, du\,.$$

Using the estimate

$$(8) \qquad e^{nu}(1 - nu^2) \leqq (1 + u)^n \leqq e^{nu}\,,$$

we have

$$\beta^{-1} \int_0^\infty e^{-t} e^{nu}(1 - nu^2)\, du \leqq A_{n+1} \leqq \beta^{-1} \int_0^\infty e^{-t} e^{nu}\, du\,.$$

Since $-t + nu = -u/\beta$, the upper bound on A_{n+1} is 1 while the lower bound is $1 - 2n\beta^2$. Hence, if $\beta = o(u^{-1})$ then $A_n \to 1$.

Proceeding to the function $f(t) = t^2$, we have

$$(P_n t^2)(x) = (1 + n\beta)^{-n} \sum_{\nu=0}^{n} \frac{\nu^2}{n^2} \binom{n}{\nu} x(x + \nu\beta)^{\nu-1}[1 - x + (n - \nu)\beta]^{n-\nu} =$$

$$= (1 + n\beta)^{-n} \sum_{\nu=0}^{n} \left[\frac{n-1}{n} \frac{\nu(\nu-1)}{n(n-1)} + \frac{\nu}{n^2}\right] \binom{n}{\nu} x(x + \nu\beta)^{\nu-1}[1 - x + (n - \nu)\beta]^{n-\nu} =$$

$$= \frac{n-1}{n} (1 + n\beta)^{-n} x\, S(2,\ n-2,\ x + 2\beta,\ 1 - x) + \frac{1}{n} (P_n t)(x)\,.$$

From the earlier work, $\frac{1}{n}(P_n t)(x) \to 0$. The other term can be written with the aid of (7) as

$$\frac{n-1}{n}(1+n\beta)^{-n}x(x+2\beta)\int_0^\infty e^{-t}\,dt\int_0^\infty e^{-s}(1+n\beta+t\beta+s\beta)^n\,ds + \tag{9}$$

$$+(n-1)(1+n\beta)^{-n}x\beta^2\int_0^\infty e^{-t}\,dt\int_0^\infty se^{-s}(1+n\beta+t\beta+s\beta)^{n-1}\,ds\,.$$

The second term of (9) is positive and may be bounded above using (8) by

$$\frac{x(n-1)\beta^2}{1+n\beta}\int_0^\infty e^{-t}\,dt\int_0^\infty se^{-s}\exp\left[\frac{t\beta+s\beta}{1+n\beta}(n-1)\right]ds = \frac{x(n-1)\beta^2}{1+n\beta}\left(\frac{1+n\beta}{1+\beta}\right)^4,$$

which tends uniformly to zero if $\beta = o(n^{-1})$. The first term of (9) can be confined by (8) to an interval

$$\frac{n-1}{n}(1+n\beta)^2x(x+2\beta)(1-6n\beta^2) < z < \frac{n-1}{n}x(x+2\beta)(1+n\beta)^2\,.$$

Thus if $\beta = o(n^{-1})$ this term tends uniformly to x^2.

§ 3. - Other polynomial operators.

Another generalization of BERNSTEIN polynomials can be obtained from another formula of JENSEN [6],

$$(x+y)(x+y+n\beta)^{n-1} = \sum_{\nu=0}^{n}\binom{n}{\nu}x(x+\nu\beta)^{\nu-1}y[y+(n-\nu)\beta]^{n-\nu-1}\,.$$

The corresponding operators Q_n are now defined by the equation

$$(Q_nf)(x) := (1+n\beta)^{1-n}\sum_{\nu=0}^{n}\binom{n}{\nu}f\left(\frac{\nu}{n}\right)x(x+\nu\beta)^{\nu-1}(1-x)[1-x+(n-\nu)\beta]^{n-\nu-1}\,.$$

It is clear that $Q_n 1 = 1$. Taking $f(t) = t$ we find that

$$(Q_n t)(x) = x(1 + n\beta)^{1-n}[S(1,\ n-1,\ x+\beta,\ 1-x) -$$

$$- \beta(n-1)\ S(1,\ n-2,\ x+\beta\ \ 1-x+\beta)] .$$

From the integral representation of S, formula (6), we see that $(Q_n t)(x) \to x$ uniformly if $\beta = o(n^{-1})$. Taking $f(t) = t^2$ we find that

$$(Q_n t^2)(x) = x(1 + n\beta)^{1-n}[S(2,\ n-2,\ x+2\beta,\ 1-x) -$$

$$-(n-2)\beta S(2,\ n-3,\ x+2\beta,\ 1-x+\beta)] ,$$

and again from the earlier work, this tends uniformly to x^2 if $\beta = o(n^{-1})$.

§ 4. - Further properties of the operators.

For the operators P_n of equation (5) it is possible to establish a generalization of VORONOWSKAJA's result about BERNSTEIN polynomials.

Theorem 2. *If f is bounded in* $[0, 1]$ *and possesses a second derivative at a point x, and if $\beta n^2 \to c$ then*

$$n[(P_n f)(x) - f(x)] \to \frac{1}{2} f''(x)[x - x^2 + 2cx^2] .$$

The proof proceeds from the equation

$$f\left(\frac{\nu}{n}\right) - f(x) = \left(\frac{\nu}{n} - x\right) f'(x) + \left(\frac{\nu}{n} - x\right)^2 \left[\frac{1}{2} f''(x) + \theta\left(\frac{\nu}{n} - x\right)\right] ,$$

from which it follows that

$$n[(P_n f)(x) - f(x)] = n\, f'(x)[(P_n(t)(x) - x] + \frac{n}{2} f''(x)\, [(P_n t^2)(x) - 2x(P_n t)(x) + x^2] +$$

$$+ n(1 + n\beta)^{-n} \sum_{\nu=0}^{n} \left(\frac{\nu}{n} - x\right)^2 \theta\left(\frac{\nu}{n} - x\right)\binom{n}{\nu} (x + \nu\beta)^{\nu-1}[1 - x + (n-\nu)\beta]^{n-\nu} .$$

From earlier estimates, we know that $n[(P_n t)(x) - x] \to 0$. If $\beta n^2 \to c$ we can show from the earlier work that

$$n[(P_n t^2)(x) - 2x(P_n t)(x) + x^2] \to x - x^2 + cx^2.$$

The last term goes to zero by an argument similar to that given in [5, p. 22].

A result of KANTOROVITCH on BERNSTEIN polynomials can also be proved for the operators P_n.

Theorem 3. *If $f(z)$ is analytic in the interior of an ellipse E with foci* 0, 1 *and if $0 \leqq \beta = o(n^{-1})$, then $(P_n f)(z) \to f(z)$ uniformly in any closed set interior to E.*

The proof will be exactly the same as in [5, p. 90] after establishing the following Lemma.

Lemma 2. *If f is a polynomial of degree $\leqq k$, then so is $P_n f$, for all n.*

Proof. We proceed by induction on k. If f is of degree $\leqq 0$, the lemma is true because $P_n 0 = 0$ and $P_n 1 = 1$. Now assume the lemma for polynomials of degree $\leqq k - 1$. Since P_n is a linear operator it will be enough if we show that $P_n f$ is of degree $\leqq k$ for the particular function

$$f(t) = t\left(t - \frac{1}{n}\right)\left(t - \frac{2}{n}\right) \cdots \left(t - \frac{k-1}{n}\right).$$

Computing in a straightforward way we find that

$$(P_n f)(x) = x\,\frac{n(n-1)\cdots(n-k+1)}{n^k}\,(1 + n\beta)^{-n}\, S(k,\ n-k,\ x + k\beta,\ 1 - x).$$

The proof will be complete if we can show that $S(k, n, x + a, 1 - x)$ is a polynomial of degree $\leqq k - 1$ for all n. That this is the case may be proved by induction on k. For $k = 1$, equation (6) shows at once that $S(1, n, x + a, 1 - x)$ is a constant. If our assertion is true for $S(k, n, x + a, 1 - x)$ then we apply the reduction formula of Lemma 1 to $S(k + 1, n, x + a, 1 - x,)$ eventually obtaining a sum of terms of the form

$$(A_\nu x + B_\nu)\ S(k,\ n - \nu,\ x + a + \nu\beta,\ 1 - x)$$

each of which, by the induction hypothesis, is a polynomial of degree $\leqq k$.

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