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A Note on the Order and Type of Integral Functions. (**)

1. – In this Note we propose to comment on the following Theorem 1 of [1] and on Theorem 2 of [2].

Theorem 1 [1]. If $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$, $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ be integral functions of the same order ϱ $(0 < \varrho < \infty)$, and types T_1 $(0 < T_1 < \infty)$ and T_2 $(0 < T_2 < \infty)$ respectively and $f(z) = \sum_{n=0}^{\infty} c_n z^n$, where $|c_n| \sim |\sqrt{a_n b_n}|$, then $f(z)$ is an integral function of order ϱ and type T such that $T \leqslant \sqrt{T_1 T_2}$.

Theorem 2 [2]. If $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ be integral functions of regular growth and of finite orders ϱ_1, ϱ_2 respectively, then the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$, where $\log(1/|c_n|) \sim \sqrt{\log(1/|a_n|)\log(1/|b_n|)}$, is an integral function of regular growth and order ϱ, such that $\sqrt{\varrho_1 \varrho_2} = \varrho$.

As pointed out by a reviewer [Math. Rev. 25 (1963), 2204, 2206] these theorems are not correct. We prove here a corrected version of these theorems.

We state the following lemma [5] without proof.

Lemma 1. *Let $L(x)$ be a positive and continuous function such that $L(kx) \sim L(x)$ as $x \to \infty$, where k is a constant $(0 < k < \infty)$. Then for every $\varepsilon > 0$, $x^{-\varepsilon} L(x) \to 0$ as $x \to \infty$.*

Theorem 1. *If $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$, $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ be integral functions of the same order ϱ $(0 < \varrho < \infty)$, and types T_1 $(0 < T_1 < \infty)$ and T_2 $(0 < T_2 < \infty)$*

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respectively, such that $|b_n| \geqslant |a_n|/L(1/|a_n|)$ *for* $n > n_0$ *and whenever* $a_n \neq 0$, *and* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where* $|c_n| \sim |\sqrt{a_n b_n}|$, *then* $f(z)$ *is an integral function of order* ϱ *and type* T *such that* $T \leqslant \sqrt{T_1 T_2}$.

Theorem 2. *If* $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *be integral functions of regular growth and finite orders* ϱ_1, ϱ_2 *respectively such that* $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ *be non-decreasing functions for* $n > n_0$, *then the function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where* $\log(1/|c_n|) \sim \sqrt{\log(1/|a_n|)\log(1/|b_n|)}$, *is an integral functions of regular growth and order* $\varrho = \sqrt{\varrho_1 \varrho_2}$.

2. – It is known [3, pp. 9, 11] that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is an integral function of finite order ϱ $(0 \leqslant \varrho < \infty)$, if and only if,

$$\mu = \limsup_{n \to \infty} \frac{n \log n}{\log(1/|c_n|)} \tag{2.1}$$

is finite and then the order ϱ of $f(z)$ is equal to μ.

Let $f(z)$ be of order ϱ $(0 < \varrho < \infty)$ and define

$$\nu = \limsup_{n \to \infty} (n\,|c_n|^{\varrho/n}). \tag{2.2}$$

If $0 < \nu < \infty$ the function $f(z)$ is of order ϱ and type T if and only if $\nu = eT\varrho$.

Lemma 2. *If* $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$, $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *be integral functions of the same order* ϱ $(0 < \varrho < \infty)$ *such that* $|b_n| \geqslant |a_n|/L(1/|a_n|)$ *for* $n > n_0$ *and whenever* $a_n \neq 0$, *where* $L(x)$ *is as in Lemma* 1, *and* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where* $|c_n| \sim |\sqrt{b_n a_n}|$, *then* $f(z)$ *is an integral function of order* ϱ.

Proof: Using (2.1) we have, for any $\varepsilon > 0$,

$$|a_n| < n^{-n/(\varrho+\varepsilon)}, \qquad n \geqslant n_1(\varepsilon),$$

$$|b_n| < n^{-n/(\varrho+\varepsilon)}, \qquad n \geqslant n_2(\varepsilon).$$

Hence, for sufficiently large n,

$$|a_n b_n| < n^{-2n/(\varrho+\varepsilon)},$$

so that

$$|\sqrt{a_n b_n}| < n^{-n/(\varrho+\varepsilon)} ,$$

$$\log |\sqrt{a_n b_n}| < - n \log n/(\varrho + \varepsilon) ,$$

$$\frac{n \log n}{\log |\sqrt{a_n b_n}|^{-1}} < \varrho + \varepsilon .$$

Hence

$$\limsup_{n\to\infty} \frac{n \log n}{\log |\sqrt{a_n b_n}|^{-1}} \leqslant \varrho$$

and, since $|c_n| \sim |\sqrt{b_n a_n}|$,

(2.3) $$\limsup_{n\to\infty} \frac{n \log n_n}{\log (1/|c_n|)} \leqslant \varrho .$$

We now prove that

(2.4) $$\limsup_{n\to\infty} \frac{n \log n}{\log (1/|c_n|)} \geqslant \varrho .$$

By hypothesis, $\varrho > 0$ and $\limsup_{n\to\infty} \dfrac{n \log n}{\log (1/|a_n|)} = \varrho$.

Therefore, there exists a sequence of $n = n_1, n_2, n_3, \ldots$ such that

$$\lim_{n\to\infty} \frac{n \log n}{\log (1/|a_n|)} = \varrho \quad \text{with } a_n \neq 0 .$$

Moreover, $|b_n| \geqslant \dfrac{|a_n|}{L(1/|a_n|)}$ for $n = n_k, n_{k+1}, \ldots$ $(n_k > n_0)$, and hence

$$|c_n|^2 = (1 + o(1)) |a_n b_n| \geqslant \frac{(1 + o(1)) |a_n|^2}{L(1/|a_n|)}$$

for $n = n_k, n_{k+1}, \ldots$.

Hence, for these n,

$$2 \log |c_n| \geqslant 2 \log |a_n| - \log L(1/|a_n|) + o(1)$$

$$= 2 \log |a_n| \left\{1 + \frac{\log L(1/|a_n|)}{2 \log (1/|a_n|)}\right\} + o(1) = 2\{1 + o(1)\} \log |a_n| + o(1)$$

since $\log L(x)/\log x \to 0$ as $x \to \infty$, by Lemma 1. Hence,

$$\log (1/|c_n|) \leqslant \{1 + o(1)\} \log (1/|a_n|) ,$$

$$\frac{n \log n}{\log (1/|c_n|)} \geqslant \frac{n \log n}{\{1 + o(1)\} \log (1/|a_n|)} \to \varrho .$$

This proves (2.4) which together with (2.3) gives

$$\varrho = \limsup_{n \to \infty} \frac{n \log n}{\log (1/|c_n|)} = \varrho(f) .$$

Proof of Theorem 1. By Lemma 2, $\varrho(f) = \varrho(f_1) = \varrho(f_2) = \varrho$. Given $\varepsilon > 0$, we have, for sufficiently large n,

$$\frac{n}{e\varrho} |a_n|^{\varrho/n} < T_1 + \varepsilon, \quad \frac{n}{e\varrho} |b_n|^{\varrho/n} > T_2 + \varepsilon ,$$

by (2.2). Hence, $\left(\frac{n}{e\varrho}\right)^2 |a_n|^{\varrho/n} |b_n|^{\varrho/n} < (T_1 + \varepsilon)(T_2 + \varepsilon),$

$$(1.1) \qquad \left(\frac{n}{e\varrho}\right) |\sqrt{a_n b_n}|^{\varrho/n} = \frac{n}{e\varrho} |a_n|^{\varrho/2n} |b_n|^{\varrho/2n} < \sqrt{(T_1 + \varepsilon)(T_2 + \varepsilon)}.$$

Since $\varrho < \infty$, $|c_n| \sim |\sqrt{a_n b_n}|$ implies $|c_n|^{\varrho/n} \sim |\sqrt{a_n b_n}|^{\varrho/n}$.

Therefore, since $\varrho(f) = \varrho$, from (1.1) we get

$$\frac{n}{e\varrho} |c_n|^{\varrho/n} = (1 + o(1)) \frac{n}{e\varrho} |\sqrt{a_n b_n}|^{\varrho/n} > \sqrt{(T_1 + \varepsilon)(T_2 + \varepsilon)} \{1 + o(1)\}$$

and hence $\limsup_{n \to \infty} \{ n |c_n|^{\varrho/n} \} = e\varrho T \leqslant e\varrho \sqrt{T_1 T_2}$ or $T \leqslant \sqrt{T_1 T_2}$.

Remark: The theorem holds if instead of

$|b_n| \geqslant \dfrac{|a_n|}{L(1/|a_n|)}$ for $n > n_0$ whenever $a_n \neq 0$, we assume $|a_n| \geqslant \dfrac{|b_n|}{L(1/|b_n|)}$ for $n > n_1$ whenever $b_n \neq 0$.

3. – Proof of Theorem 2. Since $|a_n/a_{n+1}|$ and $|b_n/b_{n+1}|$ are non decreasing for $n > n_0$, we have [4, Th. 2],

$$\liminf_{n\to\infty} \frac{n \log n}{\log (1/|a_n|)} = \lambda_1,$$

$$\liminf_{n\to\infty} \frac{n \log n}{\log (1/|b_n|)} = \lambda_2,$$

where λ_1 and λ_2 are the lower orders of $f_1(z)$ and $f_2(z)$ respectively.

Moreover, since $\lambda_1 = \varrho_1$ and $\lambda_2 = \varrho_2$ by hypothesis,

$$\lim_{n\to\infty} \frac{n \log n}{\log (1/|a_n|)} = \varrho_1,$$

$$\lim_{n\to\infty} \frac{n \log n}{\log (1/|b_n|)} = \varrho_2 .$$

Since $\log (1/|c_n|) \sim \sqrt{\log (1/|a_n|) \ \log (1/|b_n|)}$, we have

$$\sqrt{\varrho_1 \varrho_2} = \lim_{n\to\infty} \frac{n \log n}{\sqrt{\log (1/|a_n|) \log (1/|b_n|)}} = \lim_{n\to\infty} \frac{n \log n}{\log (1/|c_n|)} = \varrho .$$

Hence [4: Theorem 1, Corollary 2] $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is an integral function of regular growth and order $\sqrt{\varrho_1 \varrho_2}$.

References.

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