Riv. Mat. Univ. Parma (2) 5 (1964), 159-164

R. K. SAXENA and K. C. GUPTA (*)

On Laplace Transform.

1. - Introduction. The object of the present Note is to establish two new properties of the LAPLACE transform and to evaluate certain infinite integrals by their applications.

The conventional notation $\Phi(p) \doteqdot h(t)$ will be used to denote the LAPLACE's integral

$$\Phi(p) = p \int_0^\infty e^{-pt}\, h(t)\, dt, \tag{1.1}$$

provided that the integral is convergent and $\mathfrak{R}(p) > 0$.

The following results will be required in the sequel.

GOLDSTEIN [3, p. 105] has proved that if

$$\Phi_1(p) \doteqdot h_1(t) \quad \text{and} \quad \Phi_2(p) \doteqdot h_2(t)$$

then

$$\int_0^\infty \Phi_1(t)\, h_2(t) t^{-1}\, dt = \int_0^\infty h_1(t)\, \Phi_2(t)\, t^{-1}\, dt \tag{1.2}$$

provided the integrals are convergent.

The operational relations [1, p. 238]

$$\Gamma(\lambda + \mu)\, p\, (p + \alpha)^{-\lambda}\, (p + \beta)^{-\mu} \doteqdot t^{\lambda+\mu-1}\, e^{-\beta t}\, {}_1F_1[\mu;\ \lambda + \mu;\ (\beta - \alpha)t], \tag{1.3}$$

(*) Indirizzo: Department of Mathematics, University of Rajasthan, Jaipur, India.

where

$$\Re(\lambda + \mu) > 0, \qquad \Re(a + p) > 0$$

and

$$(1.4) \qquad -\pi^{\frac{1}{2}}\, \sigma\, p^{1-\varrho} G_{23}^{22}\left(p \,\middle|\, \begin{matrix} \frac{1}{2},\ 1 \\ \varrho,\ -\sigma,\ \sigma \end{matrix}\right) \doteqdot t^{\varrho-1}[t^{\frac{1}{2}} + (1 + t)^{\frac{1}{2}}]^{2\sigma}$$

where $\Re(\varrho) > 0$, $\Re(p) > 0$ will also be required.

2. – Theorem 1. *If*

$$\Phi(p) \doteqdot h(t)$$

and

$$\Psi(p) \doteqdot (t + \alpha)^{\lambda}\, (t + \beta)^{\mu}\, h(t),$$

then

$$(2.1) \quad \Phi(p) = \frac{p}{\Gamma(\lambda + \mu)} \int_0^{\infty} t^{\lambda+\mu-1} e^{-\beta t}\, {}_1F_1[\mu;\, \lambda + \mu;\, (\beta - \alpha)t]\, (p + t)^{-1}\, \Psi(p + t)\, dt,$$

provided the Laplace transforms of $|\, h(t)\,|$ *and* $|\,(t + \alpha)^{\lambda}\,(t + \beta)^{\mu}\, h(t)\,|$ *exist* $\Re(\lambda + \mu) > 0$, $\Re(p) > 0$ *and the integral is convergent.*

Proof. We have, by hypothesis,

$$\Psi(p) \doteqdot (t + \alpha)^{\lambda}\, (t + \beta)^{\mu}\, h(t)$$

which can also be written as

$$(2.2) \qquad p\, \frac{\Psi(p + a)}{p + a} \doteqdot e^{-at}(t + \alpha)^{\lambda}\, (t + \beta)^{\mu}\, h(t),$$

by virtue of a well-known property of (1.1).

Applying (1.2) to (1.3) and (2.2) we obtain

$$\int_0^{\infty} e^{-at}\, h(t)\, dt =$$

$$= \frac{1}{\Gamma(\lambda + \mu)} \int_0^{\infty} t^{\lambda+\mu-1}\, e^{-\beta t}\, {}_1F_1[\mu;\ \lambda + \mu;\ (\beta - \alpha)t](a + t)^{-1}\, \Psi(a + t)\, dt\,.$$

The theorem follows immediately on multiplying both sides by a and replacing a by p.

Corollary. *If*

$$\Phi(p) \doteqdot h(t)$$

and

$$\psi(p) \doteqdot t^{\lambda} (t + \beta)^{\mu} h(t),$$

then

$$(2.3) \qquad \Phi(p) = \frac{p}{\Gamma(\lambda + \mu)} \int_0^{\infty} t^{\lambda+\mu-1} {}_1F_1[\mu; \lambda + \mu; -\beta t](p + t)^{-1} \Psi(p + t) \, dt,$$

provided that the Laplace transforms of $| h(t) |$ *and* $(t + \alpha)^{\lambda} (t + \beta)^{\mu} h(t) |$ *exist,* $\Re(\lambda + \mu) > 0$ *and* $\Re(p) > 0$ *and the integral is convergent.*

When $\alpha = 0$ the theorem reduces to (2.3) on applying the KUMMER's transformation

$$e^{-z} {}_1F_1(\alpha; \varrho; z) = {}_1F_1(\alpha; \varrho; -z) .$$

On the other hand if we take $\lambda = \mu$ in (2.1) it gives a result very recently obtained by the senior author in an earlier paper [**4**, p. 183]

Example. Let [**1**, p. 294]

$$h(t) = t^{\sigma-1}(t + \alpha)^{-\lambda} \doteqdot \frac{p^{1-\sigma}\alpha^{-\lambda}}{\Gamma(\lambda)} E(\sigma, \lambda : : \alpha p) = \Phi(p),$$

where $\Re(\sigma) > 0$, $\Re(p) > 0$ and $| \arg \alpha | < \pi$, we therefore have

$$(t + \alpha)^{\lambda}(t + \beta)^{\mu}h(t) = t^{\sigma-1}(t + \beta)^{\mu}$$

$$\doteqdot \frac{p^{1-\sigma}\beta^{-\lambda}}{\Gamma(-\mu)} E(\sigma, -\mu : : \beta p) = \Psi(p),$$

where $\Re(\sigma) > 0$, $\Re(p) > 0$ and $| \arg \beta | < \pi$.

Using these values of $\Phi(p)$ and $\Psi(p)$ in (2.1) we find that

$$(2.4)\quad \begin{cases} \int_0^\infty t^{\lambda+\mu-1}(p+t)^{-\sigma} e^{-\beta t}\, {}_1F_1[\mu;\ \lambda+\mu;\ (\beta-\alpha)t]\ . \\ \qquad \cdot\ E[\sigma,\ -\mu : : \beta(p+t)]\, dt = \\ = \dfrac{\Gamma(\lambda+\mu)\Gamma(-\mu)}{\Gamma(\lambda)}\, p^{-\sigma}\alpha^{-\lambda}\beta^{-\mu}\, E\,(\sigma,\ \lambda : : \alpha p)\,, \end{cases}$$

where $\Re(\lambda+\mu)>0$, $\Re(p)>0$, $\Re(\alpha)>0$ and $\Re(\beta)>0$.

3. – Theorem 2. *If*

$$\Psi(p) \doteqdot \Phi(t)$$

and

$$p^{1-\varrho}\,[p^{\frac{1}{2}} + (1+p)^{\frac{1}{2}}]^{-2\sigma}\, \Phi(p) \doteqdot h(t)$$

then

$$(3.1)\qquad \Psi(p) = -\,p\,\pi^{\frac{1}{2}}\sigma \int_0^\infty h(t)\ G_{23}^{22}\left(p+t \,\middle|\, \begin{matrix} -\varrho-1/2,\ -\varrho \\ 0,\ -\sigma-\varrho-1,\ \sigma-\varrho-1 \end{matrix}\right) dt$$

provided that the Laplace *transforms of* $|\Phi(t)|$ *and* $|h(t)|$ *exist,* $\Re(p)>0$ *and the integral is convergent.*

Proof. By definition, we have

$$\Psi(p) = p\int_0^\infty e^{-pt}\,\Phi(t)\,dt$$

and

$$p^{1-\varrho}\,[p^{\frac{1}{2}} + (1+p)^{\frac{1}{2}}]^{-2\sigma}\,\Phi(p) = p\int_0^\infty e^{-ps}\,h(s)\,ds$$

and therefore it follows that

$$\Psi(p) = p\int_0^\infty e^{-pt}\,[t^{\varrho}\{t^{\frac{1}{2}} + (1+t)^{\frac{1}{2}}\}^{-2\sigma}\int_0^\infty e^{-st}\,h(s)\,ds]\,dt.$$

If we interchange the order of integration and evaluate the t - integral by (1.4) it gives (3.1).

The interchange of the order of integration can be easily seen to be permissib e under the conditions stated with the theorem.

Example. If we take [5]

$$p^{1-\varrho}[p^{\frac{1}{2}} + (1+p)^{\frac{1}{2}}]^{-2\sigma}\,\Phi(p) = p^{\lambda}\,[p^{\frac{1}{2}} + (1+p)^{\frac{1}{2}}]^{\,2\mu}$$
$$\doteqdot \frac{2^{2\mu}}{\Gamma(\frac{1}{2}-\lambda+\mu)}\, t^{\mu-\lambda-\frac{1}{2}}\,{}_2F_2\left[\begin{matrix} -\mu,\ \frac{1}{2}-\mu \\ 1-2\mu,\ \frac{1}{2}-\lambda+\mu\end{matrix};-t\right]$$
$$= h(t), \quad \text{where } \Re(\tfrac{1}{2}-\lambda+\mu)>0,\ \Re(p)>0\,,$$

we then have from (1.4)

$$\Phi(t) = t^{\lambda+\varrho-1}[t^{\frac{1}{2}} + (1+t)^{\frac{1}{2}}]^{2\mu+2\sigma}$$
$$\doteqdot -\pi^{\frac{1}{2}}\,(\mu+\sigma)\,G^{22}_{23}\left(p\,\middle|\,\begin{matrix} 3/2-\lambda-\varrho,\ 2-\lambda-\varrho \\ 1,\ 1-\lambda-\varrho-\mu-\varrho,\ 1-\lambda-\varrho+\mu+\sigma\end{matrix}\right)\Psi(p),$$

where $\Re(\lambda+\varrho)>0,\ \Re(p)>0$.

Applying (3.1) we find that

$$\text{(3.2)}\quad \begin{cases} \displaystyle\int_0^\infty t^{\mu-\lambda-\frac{1}{2}}\,{}_2F_2\left[\begin{matrix} -\mu,\ \frac{1}{2}-\mu \\ 1-2\mu,\ \frac{1}{2}-\lambda+\mu\end{matrix};-t\right]\cdot \\ \qquad\qquad \cdot G^{22}_{23}\left(p+t\,\middle|\,\begin{matrix} -\varrho-\frac{1}{2},\ -\varrho \\ 0,\ -\sigma-\varrho-1,\ \sigma-\varrho-1\end{matrix}\right)dt \\ = \left(\dfrac{\mu+\sigma}{\sigma}\right)2^{-2\mu}\Gamma(\tfrac{1}{2}-\lambda+\mu)\,G^{22}_{23}\left(p\,\middle|\,\begin{matrix} \frac{1}{2}-\lambda-\varrho,\ 1-\lambda-\varrho \\ 0,\ -\lambda-\varrho-\mu-\sigma,\ \mu+\sigma-\lambda-\varrho\end{matrix}\right), \end{cases}$$

where $\Re(\frac{1}{2}-\lambda+\mu)>0,\ \Re(\lambda+\varrho+1-2\mu)>0$ and $|\arg p|<\pi$.

In particular when $\lambda=2\mu$ (3.2) reduces to

$$\text{(3.3)}\quad \begin{cases} \displaystyle\int_0^\infty t^{-\mu-\frac{1}{2}}\,{}_1F_1(-\mu;\,1-2\mu;\,-t)\ G^{22}_{23}\left(p+t\,\middle|\,\begin{matrix} -\varrho-\frac{1}{2},\ -\varrho \\ 0,\ -\sigma-\varrho-1,\ \sigma-\varrho-1\end{matrix}\right)dt \\ \qquad = \left(\dfrac{\mu+\sigma}{\sigma}\right)2^{-2\mu}\Gamma(\tfrac{1}{2}-\mu)\ G^{22}_{23}\left(p\,\middle|\,\begin{matrix} \frac{1}{2}-2\mu-\varrho,\ 1-2\mu-\varrho \\ 0,\ -3\mu-\varrho-\sigma,\ \sigma-\mu-\varrho\end{matrix}\right), \end{cases}$$

where $\Re(1/2-\mu)>0,\ \Re(\varrho+1)>0$ and $|\arg p|<\pi$.

References.

[1] A. ERDELYI, ...: *Tables of integral transforms*, Vol. 1, Mc Graw-Hill, New York 1954.

[2] A. ERDELYI, ...: *Tables of integral transforms*, Vol. 2, Mc Graw-Hill, New York 1954.

[3] S. GOLDSTEIN, *Operational representation of Whittaker's confluent hypergeometric functions and Weber's parabolic cylinder functions*, Proc. London Math. Soc. (2) **34** (1932), 103-125.

[4] R. K. SAXENA, *Some infinite integrals involving E-functions*, Proc. Glasgow Math. Assoc. **5** (1962), 183-187.

[5] H. SHANKER, *On integral representations for the product of two Whittaker functions*, J. London Math. Soc. **22** (1947), 112-115.

* * *