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V. K. VARMA (*)

Some Theorems on Infinite Series wich are Self-reciprocal in a Certain Transform. (**)

1. - Introduction.

Consider the series:

(1.1) $$f(x)+f(2x)+f(3x)+\ldots,$$

(1.2) $$f(x)-f(3x)+f(5x)-f(7x)+\ldots,$$

(1.3) $$f(x)-f(3x)-f(5x)+f(7x)+f(9x)-\ldots,$$

(1.4) $$f(x)-f(5x)-f(7x)+f(11x)+f(13x)-\ldots,$$

where 1, 5, 7, 11, 13, are number prime to 6.

In a paper entitled « Self-reciprocal functions » G. N. WATSON [11] has investigated the behaviour of the individual function $f(x)$ when the series (1.1) and (1.2) are self-reciprocal in the HANKEL transform of order ν. In the same paper he proved that the series $(1/2)\Gamma(\nu)+2\sum_{n=1}^{\infty}(nx\sqrt{\pi/2})\cdot K_{\nu}(nx\sqrt{2\pi})$ is $R_{2\nu-(1/2)}$.

S. C. MITRA [6] and later on S. C. MITRA and A. SHARMA [7] have obtained some results showing the self-reciprocal property of some particular cases of the series (1.2) and (1.3) in a certain HANKEL transform. We shall get some of their results as particular cases of some general results obtained by us here.

(*) Indirizzo: Department of Mathematics, Engineering College, Bilaspur M. P., India.

The object of the paper in the first instance is to investigate the conditions under which functions involving the series mentioned above may be self-reciprocal in the generalized transform $\tilde{\omega}_{n_1, n_2, \ldots, n_n}(x)$ and to put these results in the form of some general theorems. It may be remarked that the known results may come out as particular cases of the theorems. Moreover we shall confine our approach to the series (1.2) and observe that similar results can be easily obtained for series (1.1), (1.3) and (1.4) and other series of like nature.

The kernel $\tilde{\omega}_{n_1, n_2, \ldots, n_n}(x)$ was defined ([1], (1)) as

(1.5)
$$\tilde{\omega}_{n_1, n_2, \ldots, n_n}(x) =$$
$$= \sqrt{x} \int_0^\infty \int_0^\infty \cdots \int_0^\infty J_{n_1}(t_1)\ J_{n_2}(t_2)\ \ldots\ J_{n_{n-1}}(t_{n-1})\ J_{n_n}\left(\frac{x}{t_1 t_2 \ldots t_{n-1}}\right) \frac{dt_1\, dt_2 \ldots dt_{n-1}}{t_1 t_2 \ldots t_{n-1}},$$

where the n_s can be permuted amongst themselves and $n_m + (1/2) > 0$ for $m = 1, 2, \ldots$.

A function self-reciprocal in this transform is denoted by $R_{n_1, n_2, \ldots, n_n}$. Also

(1.6)
$$\tilde{\omega}_{\nu, \nu-1}(x) = J_{2\nu-1}(2\sqrt{x}).$$

We have assumed hereafter that the series $\sum_{r=1}^{\infty} (-1)^{r-1} f((2r-1)x)$ is uniformly convergent in (α, ∞). For this we observe:

(1.7) Let $U_n(x)$ be a continuous function of x in the interval (a, b) for all values of n and let the series $\sum_{n=1}^{\infty} U_n(x)$ converge in the interval (a, b), then the points of uniform convergence of $U_n(x)$ form a set which is everywhere dense in (a, b) and has the cardinal number of the continuous ([10], p. 351).

We have frequently inverted the order of integration and summation which can be justified on the lines of H. S. Carslaw ([2], Art. 76, p. 179), viz.

(1.8) Let $U_1(x)$, $U_2(x)$, ..., $U_n(x)$ be continuous in $x \geqslant a$ and let the series $\sum_{n=1}^{\infty} U_n(x)$ converge uniformly to $f(x)$ in the arbitrary interval (a, α), where α may be taken as large as we please.

Further, let the integrals $\int_a^\infty U_1(x)\, dx$, $\int_a^\infty U_2(x)\, dx$, ... converge and the series of integrals

$$\int_a^x U_1(x)\, dx + \int_a^x U_2(x)\, dx + \ldots$$

converge uniformly in $x \geqslant a$. Then the series of integrals

$$\int_a^\infty U_1(x)\,dx + \int_a^\infty U_2(x)\,dx + \ldots$$

converges and the integral $\int_a^\infty f(x)\,dx$ converges. Also

$$\int_a^\infty f(x)\,dx = \int_a^\infty U_1(x)\,dx + \int_a^\infty U_2(x)\,dx + \ldots .$$

2. – HARDI and TITCHMARSH [4,(1)] have defined a class $A(\omega, a)$ of functions, where $0 \leqslant \omega \leqslant \pi$, $a < 1/2$. They are:

(2.1)
- (i) Analytic functions of $x = re^{i\vartheta}$, regular in the angle A defined by $r > 0$, $|\vartheta| < \omega$;
- (ii) $O(|x|^{-a-\delta})$ for small x;
- (iii) $O(|x|^{-a-1+\delta})$ for large x;

for every positive δ and uniformly in any angle $|\vartheta| \leqslant \omega - \eta < \omega$.

For this class of functions B. M. MEHROTRA [8] gave some results as follows:

If $f(x)$ belongs to R_μ and

(i) $$P(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^s\cdot\Gamma(s/2 + \mu/2 + 1/4)\cdot\Gamma(s/2 + \nu/2 + 1/4)\cdot\chi(s)\cdot x^{-s}\,ds,$$

where

$$\chi(s) = \chi(1-s)\,,$$

then

(2.2) $$g(x) = \int_0^\infty f(y)\,P(xy)\,dy$$

belongs to R_ν;

(ii)
$$K(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^s\,\Gamma(s/2+\mu/2+1/4)\cdot\Gamma(3/4+\nu/2-s/2)\cdot\chi(s)\,x^{-s}\,ds\,,$$

where $\chi(s)=\chi(1-s)$, then

$$g(x) = \frac{1}{x}\int_0^\infty f(y)\cdot K(y/x)\,dy \tag{2.3}$$

belongs to R_ν.

In particular if $f(x)$ is R_μ, then

$$g(x) = x^{\nu+1/2}\int_0^\infty \frac{y^{\mu+1/2}f(y)}{(x^2+y^2)^{[(\mu+\nu)/2]+1}}\,dy\,. \tag{2.4}$$

is R_ν.

We prove some analogous results where some of the functions involved are infinite series of the type mentioned before.

Let $f(x)$ belong to $A(\omega, a)$ and be $R_{\mu_1,\mu_2,\ldots,\mu_m}$ then ([1], (3)) it is of the form

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{ms/2}\cdot\Gamma(s/2+\mu_1/2+1/4)\ldots\Gamma(s/2+\mu_m/2+1/4)\cdot\psi(s)x^{-s}\,ds\,, \tag{2.5}$$

where $\psi(s)$ is regular in $a<\sigma<1-a$ $(s=\sigma+it)$, i.e.

$$\psi(s) \text{ is } O(e^{[(m\pi/4)-\omega+\eta]\cdot|t|})\,, \tag{2.6}$$

and satisfies $\psi(s)=\psi(1-s)$. Let us write

$$2^{ms/2}\cdot\Gamma(\mu_1/2+s/2+1/4)\ldots\Gamma(\mu_m/2+s/2+1/4)\cdot\psi(s) = P(s)\,,$$

and consider

$$F(x) = \sum_{n=1}^{\infty}(-1)^{n-1}f((2n-1)x)\,.$$

From above, we have

$$(2.7) \qquad F(x) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} (-1)^{n-1} \int_{c-i\infty}^{c+i\infty} P(s)\cdot((2n-1)x)^{-s}\, ds =$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} P(s)\cdot L(s)\cdot x^{-s}\, ds\,.$$

Let $s = c + it$ and consider the integral

$$I = \int_{-\infty}^{\infty} P(c+it)\cdot L(c+it)\cdot x^{-(c+it)}\, dt\,.$$

Putting $x = re^{i\vartheta}$, where $|\vartheta| < \omega$ and $r > 0$, we see that the modulus of the integrand does not exceed a constant multiple of

$$(2.8) \qquad r^{-c}\cdot |P(c+it)| \cdot |L(c+it)| \cdot e^{|\vartheta|\cdot|t|}\,.$$

Since by hypothesis $P(s)$ is $O(e^{(-\omega+\eta)|t|})$ uniformly in $a < \sigma << 1 - a$ as $|t| \to \infty$, the absolute value of (2.8) is not greater than

$$(2.9) \qquad \gamma^{-c}\cdot e^{-(\omega-\eta-|\vartheta|)|t|}\cdot |L(c+it)| \qquad (t > t_0)\,.$$

Hence (2.9) $\to 0$ uniformly and the integral I is uniformly and absolutely convergent in any domain of x interior to the angle A. Hence $P(s)\,L(s)$ is regular in $a < \sigma < 1 - a$, and by theorem 31 [9] $F(x)\, x^{s-1} \in L(0, \infty)$. Consequently $F(x)$ belongs to $A(\omega, a)$ and ultimately to $L^2(0, \infty)$. In fact

$$\int_0^{\infty} F(x)\, x^{s-1}\, dx = \int_0^{\infty} \sum_{n=1}^{\infty} (-1)^{n-1} f((2n-1)x) x^{s-1}\, ds = L(s) \int_0^{\infty} f(x)\, x^{s-1}\, ds\,.$$

Hence the MELLIN transform of $F(x)$ is $P(s)\,L(s)$, where $P(s)$ is the MELLIN transform of $f(x)$.

Now consider

$$2.10) \qquad \Phi(x) = \int_0^{\infty} y^{-1}\, F((x/y)\sqrt{\pi/2})\, g(y)\, dy\,,$$

where we take $g(y)$ to be $R_{\nu_1,\ldots,\nu_k,1/2}$ Substituting the MELLIN-BARNES's type integral for $F(x)$ from (2.7), we get

$$\Phi(x) = \frac{1}{2\pi i}\int_0^\infty g(y)\,\frac{\mathrm{d}y}{y}\int_{c-i\infty}^{c+i\infty} P(s)\; L(s)\left(\frac{x}{y}\sqrt{\frac{\pi}{2}}\right)^{-s}\mathrm{d}s\,. \tag{2.11}$$

Inverting the order of integration and using MELLIN's inversion formula for $g(y)$, we get

$$\Phi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} P(s)\; R(s)\; L(s)\left(x\sqrt{\frac{\pi}{2}}\right)^{-s}\mathrm{d}s, \tag{2.12}$$

where $R(s)$ is the MELLIN transform of $g(x)$.

By hypothesis $R(s)$ is of the form

$$R(s) = 2^{(K+1)s/2}\Gamma(s/2+\nu_1/2+1/4)\ldots\Gamma(s/2+\nu_k/2+1/4)\;\Gamma((s+1)/2)\;\chi(s),$$

where $\chi(s)=\chi(1-s)$, and is $O(e^{[(K+1)(\pi/4)-\omega'+\eta]|t|})$ uniformly in the strip $a<\sigma<1-a$. Hence

$$\Phi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{(m+K)s/2}\;\Gamma(\mu_1/2+s/2+1/4)\ldots\Gamma(\mu_m/2+s/2+1/4)\cdot \tag{2.13}$$

$$\cdot\Gamma(\nu_1/2+s/2+1/4)\ldots\Gamma(\nu_k/2+s/2+1/4)\;w(s)\;x^{-s}\,\mathrm{d}s\,,$$

where

$$w(s) = 2^s\,\pi^{-s/2}\Gamma((s+1)/2)\;\psi(s)\;\chi(s\,)L(s)$$

is $O(e^{[((m+K)\pi/4)-(\omega+\omega')+\eta]|t|})$ and satisfies $w(s)=w(1-s)$ by the functional equation (5) satisfied by $L(s)$, viz.

$$(2/\sqrt{\pi})^s\Gamma((s+1)/2)\;L(s) = (2/\sqrt{\pi})^{1-s}\Gamma(1-s/2)\;L(1-s)\,. \tag{2.14}$$

Hence $\Phi(x)\in A(\alpha,\,a)$, $\alpha=\omega+\omega'$ and is $R_{\mu_1,\ldots,\mu_m,\nu_1,\ldots,\nu_k}$. The inversion in the order of integration in (2.11) is justifiable under theorem 42 (TITCHMARSH **[9]**, p. 60) provided $x^{c-1}\,g(x)$ belongs to $L(0,\,\infty)$. Hence

Theorem 1. *Let*

(i) $f(x)$ *be continuous* in $x \geqslant 0$,

(ii) $\sum_{n=1}^{\infty} (-1)^{n-1} f((2n-1)x)$ *converge uniformly in* $(0, \infty)$ *to* $F(x)$,

(iii) $x^{c-1} g(x)$ *be continuous in* $(0, \infty)$ *and belong to* $L(0, \infty)$,

(iv) $f(x)$ *and* $g(x)$ *belong to* $A(\omega, a)$ *and* $f(x)$ *be its own* $R_{\mu_1,\ldots,\mu_m}$ *transform and* $g(x)$ *its own* $R_{\nu_1,\ldots,\nu_k}$. *Then*

$$\Phi(x) = \int_0^{\infty} F\left(\frac{x}{y}\sqrt{\frac{\pi}{2}}\right) \frac{g(y)}{y}\, dy$$

is $R_{\mu_1,\ldots,\mu_m,\nu_1,\ldots,\nu_k}$, *provided the integral exists.*

From (2.7) and (2.14) we have

Corollary 1. *If* $f(x)$ *is its own* $R_{n_1,\ldots,n_n,1/2}$ *transform then the series* $\sum_{n=1}^{\infty} (-1)^{n-1} f((2n-1)x\sqrt{\pi/2})$ *is* $R_{n_1,n_2,\ldots,n_n}$.

Examples.

(i) $g(x) = 2J_\mu(\sqrt{2x})\, K_\mu(\sqrt{2x})$ is $R_{\mu-(1/2),\mu+(1/2),1/2,-1/2}$ ([1], p. 66), $f(x) = x^{\nu+(1/2)} e^{-x^2/2}$ is R_ν, then

$$\Phi(x) = 2\int_0^{\infty} y^{-1} J_\mu(\sqrt{2y})\, K_\mu(\sqrt{2y}) \sum_{n=1}^{\infty} (-1)^{n-1} ((2n-1)(x/y)\sqrt{\pi/2})^{\nu+(1/2)} \cdot e^{-(1/2)((2n-1)(x/y)\sqrt{\pi/2})^2}\, dy$$

is $R_{\mu-(1/2),\mu+(1/2),\nu,-1/2}$ $(\mu \geqslant 0,\ \nu > -1/2)$.

(ii) $f(x) = J_\nu(x)$ is $R_{\nu-(1/2),\nu+(1/2)}$ ([1], p. 55), $g(x) = e^{-x}$ is $R_{1/2,-1/2}$, then

$$\Phi(x) = \int_0^{\infty} y^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} e^{-(2n-1)y\sqrt{\pi/2}} J_\nu(x/y)\, dy \tag{2.15}$$

is $R_{\nu-(1/2),\nu+(1/2),-1/2}$. On inverting the order of integration and summation, and integrating term by term, we get

$$\Phi(x) = 2\sum_{n=1}^{\infty} (-1)^{n-1} J_\nu^{1/2}((2n-1)x\sqrt{2\pi})\, K_\nu^{1/2}((2n-1)x\sqrt{2\pi}),$$

as self-reciprocal in $\tilde{\omega}_{\nu-(1/2),\nu+(1/2),-1/2}$ transform. In (2.15) if we sum the series on the right we get after a slight adyustment

$$\sqrt{x}\ \Phi(x\sqrt{2/\pi}) = \int_0^\infty y^{-3/2}\ \mathrm{sech}(1/y)\ \sqrt{xy}\ J_\nu(xy)\ dy\ .$$

Showing that $x^{-3/2}$ sech $(1/x)$ is reciprocal to $\Phi(x\sqrt{2/\pi})\sqrt{x}$ in the HANKEL transform of order ν.

From (2.15) we also find that $\Phi(x\sqrt{2/\pi})$ is the image in the LAPLACE transform of the series $x^{-1}\sum_{n=1}^{\infty}(-1)^{n-1}J_\nu((2n-1)x^{-1})$.

(iii) $f(x) = \tilde{\omega}_{\mu,\nu}(x)$ is $R_{\mu,\mu,\nu,\nu}$ ([1], (2), p. 122),

$g(x) = \sqrt{2/\pi}\sin x$ is $R_{1/2,1/2}$ ([1], (1), p. 66),

$$\Phi(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty y^{-1}\sin y\cdot\sum_{n=1}^{\infty}(-1)^{n-1}\,\tilde{\omega}_{\mu,\nu}((2n-1)(x/y)\sqrt{\pi/2})\ dy$$

is $R_{\mu,\mu,\nu,\nu,1/2}$ $(\mu > -1/2,\ \nu > -1/2)$. On inverting the order of integration and summation, we get

$$\Phi(x) = \sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^\infty \tilde{\omega}_{\mu,\nu}((2n-1)(x/y)\sqrt{\pi/2})\sin y\cdot y^{-1}\ dy =$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\,\tilde{\omega}_{\mu,\nu,1/2}((2n-1)x\sqrt{\pi/2})\ ,$$

since by definition

$$\tilde{\omega}_{\mu,\nu,\lambda}(x) = \int_0^\infty \tilde{\omega}_{\mu,\nu}(x/y)\ J_\lambda(y)\ y^{-1/2}\ dy\ .$$

Term by term integration can be justified in the following way: Consider the series of integrals

$$R_1 = \sum_{n=N}^{N'}\int_0^a \tilde{\omega}_{\mu,\nu}(x/y)\ \sin((2n-1)y)\cdot y^{-1}\ dy\ .$$

Since $\tilde{\omega}_{\mu,\nu}(x)$ is $O(x^{-1/4})$ for large x and is $O(x^{\nu+(1/2)}, x^{\mu+(1/2)})$ for small x, it can be seen that

$$\int_0^\infty \tilde{\omega}_{\mu,\nu}(x/y)\, y^{-1}\, dy$$

converges absolutely for $\nu + (1/2) > 0$, $\mu + (1/2) > 0$. Hence, by the RIEMANN-LEBESGUE's theorem ([8], p. 11), $R_1 \to 0$ as $n \to \infty$, $N' > N \geqslant N_0$, the same N_0 serving for all y in $(0, a)$, where a may be taken as large as we please. Hence the inversion is justifiable vide CARSLAW cited before.

(iv) $f(x) = I_\nu(x/2)\, K_\nu(x/2)$ is $R_{\nu-(1/2),\nu+(1/2)}$(cf. [3]), $g(x) = x\, e^{-x^2/2}$ is R_s, then

$$\Phi(x) = \int_0^\infty e^{-y^2/2} \sum_{n=1}^\infty (-1)^{n-1}\, I_\nu((n-2^{-1})xy^{-1}\sqrt{\pi/2})\, K_\nu((n-2^{-1})xy^{-1}\sqrt{\pi/2})\, dy$$

is $R_{\nu-(1/2),\nu+(1/2)}$. Let us consider a particular case of Thoerem 1, with

$$F(x) = \sum_{n=1}^\infty (-1)^{n-1}\, \tilde{\omega}_{1/2,3/2}((2n-1)x\sqrt{\pi/2})\,,$$

which is uniformly convergent in $(0, \infty)$. We known that $\tilde{\omega}_{1/2,3/2}(x)$ is $R_{1/2,1/2,3/2,3/2}$ ([1], (2), p. 122). Let $g(x)$ satisfy the

relation

$$\frac{1}{x}\, g\left(\frac{1}{x}\right) = g(x)\,,$$

then

$$\Phi(x) = \int_0^\infty y^{-1}\, g(y) \sum_{n=1}^\infty (-1)^{n-1}\, \tilde{\omega}_{1/2,3/2}((2n-1)xy^{-1}\sqrt{\pi/2})\, dy$$

is $R_{3/2,3/2,1/2}$. Writing $1/y$ for y, the R.H.S.

$$= \int_0^\infty y^{-1}\, g(1/y) \sum_{n=1}^\infty (-1)^{n-1}\, \tilde{\omega}_{1/2,3/2}((2n-1)xy\sqrt{\pi/2})\, dy =$$

$$= \sum_{n=1}^\infty (-1)^{n-1} \int_0^\infty g(y)\, \tilde{\omega}_{3/2,1/2}((2n-1)xy\sqrt{\pi/2})\, dy\,.$$

The inversion in the order of integration and summation can be justified, since we take $g(x)$ to be continuous and $\in L(0, \infty)$. Now let us denote the LAPLACE transform symbolically as

$$g(p) \doteqdot h(t), \tag{2.16}$$

then

$$\Phi(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^{\infty} y\, \tilde{\omega}((2n-1)xy\sqrt{\pi/2})\, dy \int_0^{\infty} e^{-yt}\, h(t)\, dt =$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^{\infty} h(t)\, dt \int_0^{\infty} y\, e^{-yt}\, \omega_{3/2,1/2}((2n-1)xy\sqrt{\pi/2})\, dy =$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} (2n-1)x \sqrt{\frac{\pi}{2}} \int_0^{\infty} e^{-(2n-1)xt^{-1}\sqrt{\pi/2}}\, h(t) \cdot t^{-3}\, dt\,.$$

Since $x\, e^{-x}$ is self-reciprocal w.r. to the kernel $\omega_{3/2,1/2}(x)$ and we assume the integrals to be absolutely convergent. Further let

$$t\, h\left(\frac{1}{t}\right) \doteqdot \chi(p)\,, \tag{2.17}$$

then we have

$$\Phi(x) = \sum_{n=1}^{\infty} (-1)^{n-1}\, \chi((2n-1)x\sqrt{\pi/2}) \tag{2.18}$$

is $R_{3/2,3/2,1/2}$ provided the series converges, and $\chi(x)$ is integrable in $(0, \infty)$ and $\to 0$ as $x \to \infty$. Hence

Theorem 1A. *Let*

(i) *$g(y)$ be continuous and belong to $L(0, \infty)$ and satisfy the relation $(1/y) \cdot g(1/y) = g(y)$.*

(ii) *$g(p) \doteqdot h(t)$, where $t \cdot h(1/t)$ is $O(t^{\delta}) \in L(0, \alpha)$ and $e^{-pt}\, h(t) \in L(0, \infty)$.*

(iii) $t\,h(1/t) \doteqdot \chi(p)$, *where* $\chi(p)$ *is continuous and integrable in* $(0, \infty)$, *and* $\to 0$ *as* $p \to \infty$; *then*

$$\Phi(y) = \int_0^\infty y^{-1} g(y) \sum_{n=1}^\infty (-1)^{n-1}\, \tilde{\omega}_{3/2,1/2}((2n-1)xy^{-1}\sqrt{\pi/2})\,\mathrm{d}y =$$

$$= \sum_{n=1}^\infty (-1)^{n-1}\, \chi((2n-1)x\sqrt{\pi/2})$$

is $R_{3/2,3/2,1/2}$.

Examples.

(i) $g(y) = y/(1+y)^3 = (1/y)\,g(1/y)$,

$g(p) = p/(1+p)^3 \doteqdot e^{-x}x^2 = h(x)$,

$e^{-1/x}x^{-1} \doteqdot 2p\,K_0(2\sqrt{p}) = \chi(p)$,

hence

$$\int_0^\infty \frac{1}{(1+y)^3} \sum_{n=1}^\infty (-1)^{n-1}\, \tilde{\omega}_{3/2,1/2}((2n-1)xy^{-1}\sqrt{\pi/2})\,\mathrm{d}y =$$

$$= 2\sum_{n=1}^\infty (-1)^{n-1}\,((2n-1)x\sqrt{\pi/2})\;K_0(2\sqrt{(2n-1)\,x\sqrt{\pi/2}}),$$

is $R_{3/2,3/2,1/2}$.

(ii) $g(y) = y^2/(1+y^2)^{5/2} = (1/y)\,g(1/y)$,

$g(p) = p^2/(1+p^2)^{5/2} \doteqdot (1/3)t^2 J_1(t) = h(t)$,

$t\,h(1/t) = (1/3)t^{-1} J_1(1/t) \doteqdot (2/3)p\,J_1(\sqrt{2p})\,K_1(\sqrt{2p})$,

then

$$\int_0^\infty \frac{y}{(1+y^2)^{5/2}} \sum_{n=1}^\infty (-1)^{n-1}\, \tilde{\omega}_{3/2,1/2}((2n-1)xy^{-1}\sqrt{\pi/2})\,\mathrm{d}y =$$

$$= \frac{2}{3}\sum_{n=1}^\infty (-1)^{n-1}\,(2n-1)x\sqrt{\pi/2}\;J_1(\sqrt{(2n-1)x\sqrt{2\pi}})\;K_1(\sqrt{(2n-1)x\sqrt{2\pi}})$$

is $R_{3/2,3/2,1/2}$.

References.

[1] K. P. BHATNAGAR: (1) *On certain theorems on self-reciprocal functions*, Acad. Roy. Belgique, Bull. Cl. Sci. (5) **39** (1953), 42-69; (2) *On self-reciprocal functions*, Ganita **4** (1953), 19-37; (3) *Two theorems on self-reciprocal functions and a new transform*, Bull. Calcutta Math. Soc. **45** (1953), 109-112.

[2] H. S. CARSLAW: *Introduction to the theory of Fourier's series and integrals*, Macmillan, London 1930.

[3] A. ERDELYI: *On some expansions in Laguerre polynomials*, J. London Math. Soc. **13** (1938), 154-156.

[4] G. H. HARDY and E. C. TITCHMARSH: (1) *Self-reciprocal functions*, Quarterly J. of Math. (Oxford series) **1** (1930), 196-231; (2) *Formulae connecting different classes of self-reciprocal functions*, Proc. London Math. Soc. (2) **33** (1931), 225-232.

[5] C. J. MAHNSTEN: Crelle's J. **38** (1849), 1-39.

[6] S. C. MITRA: *On certain self-reciprocal functions*, Bull. Calcutta Math. Soc. **41** (1949), 1-5.

[7] S. C. MITRA and A. SHARMA: *On certain self-reciprocal functions*, Ganita **1** (1950), 31-38.

[8] B. M. MEHROTRA: (1) *Some theorems on self-reciprocal functions*, Proc. London Math. Soc. (2) **34** (1932), 231-240; (2) *Definite integrals involving self-reciprocal functions*, Bull. Calcutta Math. Soc. **24** (1933), 163-176.

[9] E. C. TITCHMARSH: *Introduction to the theory of Fourier integrals*, Clarendon, Oxford 1937.

[10] TOWNSEND: *Real variables*.

[11] G. N. WATSON: *Some self-reciprocal functions*, Quarterly J. of Math. (Oxford series) **2** (1931), 298-309.

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