Riv. Mat. Univ. Parma (2) 5 (1964), 211-218

R. K. Srivastava (*)

On the Type of Integral Functions Represented by Dirichlet Series.

1. - Let $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ be an integral function, where $s = \sigma + it$, $0 < \lambda_1 < \lambda_2 < \dots$, $\lim_{n\to\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} (\log n)/\lambda_n = 0$. Let σ_c and σ_a denote the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$. If $\sigma_c = \sigma_a = \infty$, $f(s)$ defines an integral function.

In this paper we shall obtain the relations between two or more integral functions and study the relations between their types. The results are given in the form of theorems.

2. - Theorem 1. *Let* $f_1(s) = \sum_{n=1}^{\infty} a_{1,n} e^{s\lambda_{1,n}}$ *and* $f_2(s) = \sum_{n=1}^{\infty} a_{2,n} e^{s\lambda_{2,n}}$ *be integral functions of the same finite non-zero linear order* ϱ, *finite non-zero types* T_1 *and* T_2, *lower types* t_1 *and* t_2 *respectively and* $\limsup_{n\to\infty} \{(\log n)/\lambda_{1,n}\} = 0$. *Then the function* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$, *where*

(i) $$\lambda_{1,n} \sim \lambda_{2,n} \sim \lambda_n ,$$

(ii) $$|a_n| \sim |\sqrt{|a_{1,n}|\,|a_{2,n}|}|$$

(*) Indirizzo: Department of Mathematics and Astronomy, Lucknow University, Lucknow, India.

is an integral function such that

$$T \leqslant \sqrt{T_1 T_2} \tag{2.1}$$

provided $f(s)$ is of linear order ϱ and type T. Further, if

(iii) $$\lambda_{1,n} \sim \lambda_{1,n+1}\,,$$

(iv) $\dfrac{\log|a_{1,n}/a_{1,n+1}|}{\lambda_{1,n+1}-\lambda_{1,n}}$ *and* $\dfrac{\log|a_{2,n}/a_{2,n+1}|}{\lambda_{2,n+1}-\lambda_{2,n}}$ *be non-decreasing functions of n for large n, then*

$$t \geqslant \sqrt{t_1\, t_2}\,, \tag{2.2}$$

were the function $f(s)$ is of linear order ϱ and linear lower type t.

Proof. The integral function $g(s) = \sum_{p=1}^{\infty} b_p\, e^{s\lambda_p}$, where $\limsup_{p\to\infty}\{(\log p)/ /\lambda p\} = 0$, is of finite non-zero linear order ϱ and type T, if and only if ([3], p. 72)

$$\limsup_{p\to\infty}\{(\lambda_p/e\varrho)\,|\,b_p\,|^{\varrho/\lambda_p}\} = T\,. \tag{2.3}$$

Using (2.3) for the functions $f_1(s)$ and $f_2(s)$, we get for sufficiently large n and $\varepsilon > 0$

$$(\lambda_n/e\varrho)\,|\,a_{1,n}\,|^{\varrho/\lambda_n} < T_1 + \varepsilon, \quad (\lambda_n/e\varrho)\,|\,a_{2,n}\,|^{\varrho/\lambda_n} < T_2 + \varepsilon\,,$$

since $\lambda_{1,n} \sim \lambda_{2,n} \sim \lambda_n$.

Again, T_1 and T_2 are finite, therefore,

$$(\lambda_n/e\varrho)\,|\sqrt{|\,a_{1,n}\,|\,|\,a_{2,n}\,|}\,|^{\varrho/\lambda_n} < \sqrt{T_1 T_2} + o(1)\,. \tag{2.4}$$

Now we know the following lemma ([1]).
If $f(x) \sim g(x)$ as $x \to \infty$, then for any finite constant K and $x > 0$

$$\{f(x)\}^{K/x} \sim \{g(x)\}^{K/x}\,.$$

Since

$$|\,a_n\,| \sim |\sqrt{|\,a_{1,n}\,|\,|\,a_{2,n}\,|}\,|\,,$$

therefore,

$$(2.5) \qquad | a_n |^{\varrho/\lambda_n} \sim | \sqrt{| a_{1,n} | | a_{2,n} |} |^{\varrho/\lambda_n}.$$

Now (2.1) follows, on using (2.4) and (2.5).
Again, we know that ([2])

$$(2.6) \qquad \liminf_{p\to\infty} \{ (\lambda_p/e\varrho) | b_p |^{\varrho/\lambda_p} \} = t ,$$

where t is the linear lower type of $g(s)$, provided $\limsup_{p\to\infty} \{ (\log p)/\lambda_p \} = 0$, $\lambda_p \sim \lambda_{p+1}$ and $\dfrac{\log | b_p/b_{p+1} |}{\lambda_{p+1} - \lambda_p}$ is a non-decreasing function of p for large p.

Therefore, using (2.6) for the functions $f_1(s)$ and $f_2(s)$, we can similarly deduce that $t \geqslant \sqrt{t_1 t_2}$.

Corollary 1. For $m = 1, 2, \ldots , p$ let $f_m(s) = \sum_{n=1}^{\infty} a_{m,n} \, e^{s\lambda_{m,n}}$ denotes an integral function of finite non-zero linear type T_m lower type t_m, order ϱ and $\limsup_{n\to\infty} \{ (\log n)/\lambda_{m,n} \} = 0$. Then the function $f(s) = \sum_{n=1}^{\infty} a_n \, e^{s\lambda_n}$, where $\lambda_{m,n} \sim \lambda_n$ and $| a_n | \sim | \sqrt[p]{| a_{1,n} | | a_{2,n} | \ldots | a_{p,n} |} |$, is an integral function such that $T \leqslant \leqslant \sqrt[p]{T_1 T_2 \ldots T_p}$ provided $f(s)$ is of linear order ϱ and type T. Further, if $\lambda_{m,n} \sim \lambda_{m,n+1}$ and $\{ \log | a_{m,n}/a_{m,n+1} | \}/\{ \lambda_{m,n+1} - \lambda_{m,n} \}$ is a non-decreasing function of n for large n, then $t \geqslant \sqrt[p]{t_1 t_2 \ldots t_p}$, where the function $f(s)$ is of linear order ϱ and lower type t.

Corollary 2. If in Corollary 1, we take the integral functions $f_m(s)$ to be of perfectly linear regular growth, then $f(s)$ is an integral function of perfectly linear regular growth such that $T = \sqrt[p]{T_1 T_2 \ldots T_p}$.

Theorem 2. *Let* $f_1(s) = \sum_{n=1}^{\infty} a_{1,n} \, e^{s\lambda_{1,n}}$ and $f_2(s) = \sum_{n=1}^{\infty} a_{2,n} \, e^{s\lambda_{2,n}}$ *be integral functions of finite non-zero linear regular growths* ϱ_1, ϱ_2, *types* T_1, T_2, *lower types* t_1, t_2 *respectively and* (i) $\limsup_{n\to\infty} \{ (\log n)/\lambda_{1,n} \} = 0$, (ii) $\dfrac{\log | a_{1,n}/a_{1,n+1} |}{\lambda_{1,n+1} - \lambda_{1,n}}$, $\dfrac{\log | a_{2,n}/a_{2,n+1} |}{\lambda_{2,n+1} - \lambda_{2,n}}$ *be non-decreasing functions of* n *for large* n. *Then the function* $f(s) = \sum_{n=1}^{\infty} a_n \, e^{s\lambda_n}$, *where*

$$| a_n | \sim | a_{1,n} | | a_{2,n} |, \qquad \lambda_n \sim \lambda_{1,n} \sim \lambda_{2,n} \sim \lambda_{n+1},$$

is an integral function of regular growth such that

$$\alpha = \alpha_1 + \alpha_2 , \tag{2.7}$$

$$\left(\frac{T}{\alpha}\right)^{\alpha} \leqslant \left(\frac{T_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{T_2}{\alpha_2}\right)^{\alpha_2}, \tag{2.8}$$

$$\left(\frac{t}{\alpha}\right)^{\alpha} \geqslant \left(\frac{t_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{t_2}{\alpha_2}\right)^{\alpha_2}, \tag{2.9}$$

where $\alpha = 1/\varrho$, $\alpha_1 = 1/\varrho_1$, $\alpha_2 = 1/\varrho_2$ *and* ϱ, T, t *are respectively the linear order, type and lower type of* $f(s)$.

Proof: For an integral function $g(s) = \sum_{p=1}^{\infty} b_p e^{s\lambda_p}$, where $\limsup_{p\to\infty} \{ (\log p)/ /\lambda_p \} = 0$ to be of finite linear order ϱ, it is necessary and sufficient that ([3], p. 69)

$$\liminf_{p\to\infty} \frac{\log (1/|b_p|)}{\lambda_p \log \lambda_p} = 1/\varrho . \tag{2.10}$$

Further, if $g(s)$ is of linear regular growth, (2.10) reduces to

$$\lim_{p\to\infty} \frac{\log (1/|b_p|)}{\lambda_p \log \lambda_p} = 1/\varrho. \tag{2.11}$$

Thus, using (2.11) for the functions $f_1(s)$ and $f_2(s)$, we obtain

$$\lim_{n\to\infty} \frac{\log (1/|a_{1,n}|)}{\lambda_{1,n} \log \lambda_{1,n}} = 1/\varrho_1, \quad \lim_{n\to\infty} \frac{\log (1/|a_{2,n}|)}{\lambda_{2,n} \log \lambda_{2,n}} = 1/\varrho_2 .$$

Therefore,

$$\lim_{n\to\infty} \frac{\log (1/|a_n|)}{\lambda_n \log \lambda_n} = (1/\varrho_1) + (1/\varrho_2) ,$$

since $|a_n| \sim |a_{1,n}|\,|a_{2,n}|$ and $\lambda_n \sim \lambda_{1,n} \sim \lambda_{2,n}$. Thus $f(s)$ is an integral function of regular growth ϱ such that

$$1/\varrho = (1/\varrho_1) + (1/\varrho_2) .$$

Again, $|a_n| \sim |a_{1,n}|\,|a_{2,n}|$ and $\lambda_n \sim \lambda_{1,n} \sim \lambda_{2,n}$. Therefore, using the lemma ([1]), we have

$$\lambda_n^{\alpha} |a_n|^{1/\lambda_n} \sim \lambda_n^{\alpha_1+\alpha_2}(|a_{1,n}|\,|a_{2,n}|)^{1/\lambda_n}$$

$$< (1+\varepsilon)[\lambda_{1,n}^{\alpha_1} |a_{1,n}|^{1/\lambda_{1,n}} \times \lambda_{2,n}^{\alpha_2} |a_{2,n}|^{1/\lambda_{2,n}}] , \ \varepsilon > 0 .$$

Hence, using (2.3) and taking limit on both the sides as $n \to \infty$, we get

$$\left(\frac{T}{\alpha}\right)^{\alpha} \leqslant \left(\frac{T_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{T_2}{\alpha_2}\right)^{\alpha_2} .$$

Similarly, it can be shown that

$$\left(\frac{t}{\alpha}\right)^{\alpha} \geqslant \left(\frac{t_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{t_2}{\alpha_2}\right)^{\alpha_2} .$$

Corollary 1. For $m = 1, 2, \ldots, p$, let $f_m(s) = \sum_{n=1}^{\infty} a_{m,n} e^{s\lambda_{m,n}}$ denotes an integral function of finite non-zero linear regular growth ϱ_m, type T_m, lower type t_m, such that $\dfrac{\log |a_{m,n}/a_{m,n+1}|}{\lambda_{m,n+1} - \lambda_{m,n}}$ is a non-decreasing function of n for large n and $\limsup_{n\to\infty} \{ (\log n)/\lambda_{m,n} \} = 0$. Then the function $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$, where

$$|a_n| \sim |a_{1,n}|\,|a_{2,n}| \ldots |a_{p,n}| , \qquad \lambda_n \sim \lambda_{m,n} \sim \lambda_{n+1} ,$$

is an integral function of linear regular growth such that

$$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_p ,$$

$$\left(\frac{T}{\alpha}\right)^{\alpha} \leqslant \left(\frac{T_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{T_2}{\alpha_2}\right)^{\alpha_2} \cdots \left(\frac{T_p}{\alpha_p}\right)^{\alpha_p} ,$$

and

$$\left(\frac{t}{\alpha}\right)^{\alpha} \geqslant \left(\frac{t_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{t_2}{\alpha_2}\right)^{\alpha_2} \cdots \left(\frac{t_p}{\alpha_p}\right)^{\alpha_p},$$

where $\alpha = 1/\varrho$, $\alpha_m = 1/\varrho_m$ and ϱ, T, t are respectively the linear order type and lower types of $f(s)$.

Corollary 2. If in the above corollary all the functions are of perfectly linear regular growth, then so is $f(s)$ and

$$\left(\frac{T}{\alpha}\right)^{\alpha} = \left(\frac{T_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{T_2}{\alpha_2}\right)^{\alpha_2} \cdots \left(\frac{T_p}{\alpha_p}\right)^{\alpha_p}.$$

Theorem 3. *Let* $f_1(s) = \sum_{n=1}^{\infty} a_{1,n} e^{s\lambda_{1,n}}$ *and* $f_2(s) = \sum_{n=1}^{\infty} a_{2,n} e^{s\lambda_{2,n}}$ *be integral functions of finite non-zero linear regular growths* ϱ_1, ϱ_2, *types* T_1, T_2 *and lower types* t_1, t_2 *respectively such that*

(i) $$\limsup_{n\to\infty} \{ (\log n)/\lambda_{1,n} \} = 0 ,$$

(ii) $$\lambda_{1,n} \sim \lambda_{1,n+1},$$

(iii) $\dfrac{\log |a_{1,n}/a_{1,n+1}|}{\lambda_{1,n+1} - \lambda_{1,n}}$, $\dfrac{\log |a_{2,n}/a_{2,n+1}|}{\lambda_{2,n+1} - \lambda_{2,n}}$ *be non-decreasing functions of n and n large.*

Then the function

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} ,$$

where

$$|a_n| \sim |a_{1,n}| \, |a_{2,n}| , \qquad \lambda_n \sim \lambda_{1,n} \sim \lambda_{2,n}$$

is an integral function of regular growth such that

$$\left(\frac{t}{\alpha}\right)^{\alpha} \leqslant \frac{\left(\frac{T_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{t_2}{\alpha_2}\right)^{\alpha_2}}{\left(\frac{t_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{T_2}{\alpha_2}\right)^{\alpha_2}} \leqslant \left(\frac{T}{\alpha}\right)^{\alpha}, \tag{2.12}$$

where $\alpha = 1/\varrho$, $\alpha_1 = 1/\varrho_1$, $\alpha_2 = 1/\varrho_2$ *and* ϱ, T, t *are respectively the linear order, type and lower type of* $f(s)$.

Proof: We know that for non-negative functions $f(x)$ and $g(x)$

$$\underline{\lim}_{x\to\infty}\{f(x)\cdot g(x)\} \leqslant \begin{matrix} \overline{\lim}_{x\to\infty} f(x) \times \underline{\lim}_{x\to\infty} g(x) \\ \\ \underline{\lim}_{x\to\infty} f(x) \times \overline{\lim}_{x\to\infty} g(x) \end{matrix} \leqslant \overline{\lim}_{x\to\infty}\{f(x)\cdot g(x)\}.$$

Since $|a_n| \sim |a_{1,n}|\,|a_{2,n}|$ and $\lambda_n \sim \lambda_{1,n} \sim \lambda_{2,n}$, therefore,

$$\underline{\lim}_{n\to\infty}\{\lambda_n^{\alpha}|a_n|^{1/\lambda_n}\} \leqslant \begin{matrix} \overline{\lim}_{n\to\infty} \lambda_{1,n}^{\alpha_1}|a_{1,n}|^{1/\lambda_{1,n}} \times \underline{\lim}_{n\to\infty} \lambda_{2,n}^{\alpha_2}|a_{2,n}|^{1/\lambda_{2,n}} \\ \\ \underline{\lim}_{n\to\infty} \lambda_{1,n}^{\alpha_1}|a_{1,n}|^{1/\lambda_{1,n}} \times \overline{\lim}_{n\to\infty} \lambda_{2,n}^{\alpha_2}|a_{2,n}|^{1/\lambda_{2,n}} \end{matrix} \leqslant \overline{\lim}_{n\to\infty}\{\lambda_n^{\alpha}|a_n|^{1/\lambda_n}\}.$$

The results follow from the above inequalities, because the limits exist due to (2.3) and (2.6).

Theorem 4. *The integral functions* $f_1(s) = \sum_{n=1}^{\infty} a_{1,n}\, e^{s\lambda_{1,n}}$ *and* $f_2(s) = \sum_{n=1}^{\infty} a_{2,n}\, e^{s\lambda_{2,n}}$ *are of the same finite non-zero perfectly linear regular growth if, and only if,*

$$\log\frac{|a_{1,n}|}{|a_{2,n}|} = o(\lambda_{1,n}),$$

for large n, *provided*

(i) $f_1(s)$ *and* $f_2(s)$ *are of the same finite non-zero linear order* ϱ,

(ii) $$\lim_{n\to\infty}\sup\{(\log n)/\lambda_{1,n}\} = 0,$$

(iii) $$\lambda_{1,n} \sim \lambda_{2,n} \sim \lambda_{2,n+1}$$

and

(iv) $\dfrac{\log|a_{1,n}/a_{1,n+1}|}{\lambda_{1,n+1}-\lambda_{1,n}}$, $\dfrac{\log|a_{2,n}/a_{2,n+1}|}{\lambda_{2,n+1}-\lambda_{2,n}}$ *are non-decreasing functions of* n *for large* n.

P r o o f : If the given functions $f_1(s)$ and $f_2(s)$ be each of finite perfectly linear regular growth T, we have on using (2.3) and (2.6)

$$\lim_{n\to\infty}\{(\lambda_{1,n}/e\varrho)\mid a_{1,n}\mid^{\varrho/\lambda_{1,n}}\} = T = \lim_{n\to\infty}\{(\lambda_{2,n}/e\varrho)\mid a_{2,n}\mid^{\varrho/\lambda_{2,n}}\}. \tag{2.13}$$

Therefore,

$$\lim_{n\to\infty}\{(\varrho/\lambda_{1,n})(\log\mid a_{1,n}\mid - \log\mid a_{2,n}\mid)\} = 0\,,$$

since

$$\lambda_{1,n} \sim \lambda_{2,n}\,.$$

Hence

$$\log\frac{\mid a_{1,n}\mid}{\mid a_{2,n}\mid} = o(\lambda_{1,n}) \tag{2.14}$$

for large n. Further, if $f_1(s)$ and $f_2(s)$ be of finite perfectly linear regular growths T_1 and T_2 respectively, we get on using (2.13)

$$\log T_1 - \log T_2 = \lim_{n\to\infty}\{(\varrho/\lambda_{1,n})(\log\mid a_{1,n}\mid - \log\mid a_{2,n}\mid)\} = 0\,,$$

if (2.14) holds. Hence $T_1 = T_2$.

In conclusion I offer my grateful thanks to Dr. S. K. BOSE for his kind help in the preparation of this paper.

References.

[1] R. S. L. SRIVASTAVA, *On the order and type of integral functions*, Riv. Mat. Univ. Parma **10** (1959), 249-255.

[2] R. K. SRIVASTAVA, *On the coefficient of an integral function represented by Dirichlet series of finite order*, Ganita **13** (1962), 25-35.

[3] Y. C. YU, *Sur les droites de Borel de certaines fonctions entières*, Ann. Sci. École Norm. Sup. (3) **68** (1951), 65-104.

* * *