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The Impossibility of Rings with Dual Distributive Laws. (**)

1. - Introduction.

Besides the customary pair of twin distributive laws in a ring, we may conceive of a dual distributive law in which « addition » is distributive with respect to «multiplicatior ». It is shown in the current Note that a ring which is subject to the dual law is necessarily trivial, i.e. reduces to the single element « zero ».

In the proof of this result, two of the elements appearing in the formal expression for the new law are assigned the value « zero » at certain critical stages. Since «zero » is a special element within a ring, with respect to «multiplication » as well as its own operation «addition », we also consider a modified dual distributive law in which the two elements in question are barred from assuming the value «zero ». Even under this weakened form of the new law, the rings turn out to be trivial. The sole exceptions are a certain class of 2-element rings.

Upon reexamining the proofs of the two theorems, we observe that the associative law for « multiplication » was never used and that it suffices to assume that «addition » is subject to a cancellation law ($x + a = y + a \Longrightarrow x = y$) instead of insisting on « additive » inverses. (The « zero » element must be retained, however.) The theorem therefore applies to a much wider class of algebraic structures than rings, i.e. to the class of « peudo-rings ».

Summing up, we note that there is one important respect in which pseudo-rings (including rings) differ from lattices. A lattice, because of the principle of duality, posseses either no or two distributive laws; a pseudo-ring is characterized by exactly one pair of twin distributive laws.

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2. - Proof of the first theorem.

The elements of the ring form an Abelian group with respect to the first operation or associative binary composition (« + ») and a semi-group with respect to the second associative binary composition (« . »). The « . » is distributive with respect to the « + ». See [1]. Except for incisiveness, we follow the usual convention and represent $u \cdot v$ by uv.

Now let a, x, y by *any* three (arbitrary) elements of our ring. If it is assumed that the « + » is distributive with respect to the « . », then

$$\begin{aligned} a + (x \cdot y) &= (a + x) \cdot (a + y) \\ &= a \cdot (a + y) + x \cdot (a + y) \\ &= a^2 + ay + xa + xy\,, \end{aligned} \tag{2.1}$$

where the distributive law of the « . » with respect to the « + » has been applied in the final steps. I.e.

$$a = a^2 + ay + xa\,. \tag{2.2}$$

Since x and y are arbitrary, we can set $x = y = 0$ in (2.2), where « 0 » is the identity element in our Abelian group, and obtain

$$a^2 = a\,, \tag{2.3}$$

i.e. every element in the ring is idempotent with respect to the « . ». Now we revert to (2.2) and set $x = a$ and $y = 0$, discovering that

$$a^2 = 0\,. \tag{2.4}$$

[We have used the deducible relation $0 \cdot w = w \cdot 0 = 0$ in arriving at (2.3) and (2.4).] I.e.

$$a = 0 \tag{2.5}$$

and every element in our ring coincides with « 0 ».

Nowhere in our demonstration have we used the property $au = av \Longrightarrow u = v$, which characterizes integral domains and fields but which does not describe the more general type of ring.

3. - A modified distributive law, etc..

The preceding demonstration was based largely on letting x and (or) y assume the value « 0 » at certain stages. But « 0 » is an exceptional element in a ring («0» is self-preserving under the « · », i. e. $0 \cdot a = a \cdot 0 = 0$, and the elements of a field, « 0 » *excluded*, form a group with respect to the « · ») and we may choose to exclude « 0 » from certain phases of our dual distributive law. Specifically we may restrict the x and y in (2.1), which expresses the dual law, to values other than « 0 », while continuing to allow a to assume all values including «0». We are then dealing with a modified dual distributive law. *Our modified theorem states that the « + » cannot be distributive in this modified fashion with respect to the « · » in any ring consisting of three or more elements.* It requires, of course, a fresh proof, which proceeds as follows.

Since x and y are arbitrary (except for the noted restriction), (2.2) implies

$$(3.1) \qquad a = a^2 + ax + xa \qquad (x \neq 0)$$

and

$$(3.2) \qquad a = a^2 + ay = ya \qquad (y \neq 0)\,.$$

« Subtracting » (3.2) from (3.1) and noting that $-(ay) = a(-y)$ yields

$$(3.3) \qquad a(x-y) + (x-y)a = 0 \qquad (x,\ y \neq 0).$$

Now given *any* element z, we may pick an $x \neq 0,\ z$ and then solve the equation

$$(3.4) \qquad x + (-y) = z \qquad (x \neq 0,\ z)$$

for y. (Our choice of x compels the ring to contain at least three distinct elements, a requirement already accounted for in the statement of the theorem.) The solution $y \neq 0$ since $x - 0 = x \neq z$. Because such a y exists (3.3) reduces to

$$(3.5) \qquad az + za = 0 \qquad (a,\ z \text{ arbitrary})\,.$$

Now (3.5) and (3.1) imply

$$(3.6) \qquad a^2 = a\,,$$

which is identidal with (2.3). Also (3.6), (2.2) and (3.5) imply

$$(3.7) \qquad 0 = ay + xa = ay + (-ax) = a(y-x) \qquad (x,\ y \neq 0)\,.$$

Now given any a, we may take $y \neq a,\ 0$ and solve the equation

$$(3.8) \qquad y - x = a$$

for x (once again, the solution $x \neq 0$). The existence of such an x, (3.7) and (3.6) lead to

$$0 = a^2 = a \qquad (a \text{ arbitrary}), \tag{3.9}$$

which contradicts our assumption that the ring contains et least three distinct elements.

4. - Two-element rings.

Since a ring with one element is trivial and the « + » cannot be either qualifiedly or unqualifiedly distributive with respect to the « · » in a ring containing three or more elements, the only possibility which offers any promise is a ring with two distinct elements. The elements of such a ring may be represented by $(0, \alpha)$, and we may have

$$\text{either} \qquad \alpha^2 = \alpha \cdot \alpha = 0 \qquad \text{or} \qquad \alpha^2 = \alpha. \tag{4.1}$$

(The second possibility implies that (α) is a group under the « · ».) If we set $a, x, y = \alpha$ in (2.2), this equation reduces to

$$\alpha = 0 \tag{4.2}$$

when $\alpha^2 = 0$, an obviously false result. The same substitution reduces (2.2) to

$$\alpha = \alpha + \alpha + \alpha, \qquad \text{i.e.} \qquad \alpha + \alpha = 0 \tag{4.3}$$

when $\alpha^2 = \alpha$. (4.3) is correct since $(0, \alpha)$ is a group under the « + » and α is its own inverse with respect to the « + ». We conclude that the 2-element rings (they are all isomorphic) characterized by $\alpha^2 = \alpha$ obey the modified distribution law of section **3**. They are inconsistent with the *un* modified law conveyed by (2.1) since $a, x = \alpha$; $y = 0$ reduces (2.2) to

$$\alpha = \alpha^2 + \alpha^2 \quad \Longrightarrow \quad \alpha = 0 \tag{4.4}$$

when $\alpha^2 = \alpha$ and α is distinct from « 0 » by assumption.

5. - Extension to pseudo-rings.

It will be observed that the associative law for the « · » operation, normally assumed for a ring, was not used anywhere in sections **2** and **3**. It will also be seen that the passage from (2.1) to (2.2) does not depend inescapably on the existence of « additive » inverses but can be validated by assuming an « additive » (« + ») cancellation law (a much weaker assumption). The relation $w \cdot 0 = 0$, used in arriving at (2.3) and (2.4), can be established without postulating the existence of « additive » inverses. The result of section **2** therefore applies to a much wider class of algebraic structures than rings, i.e. to speudo-rings. (This was also suggested by Dr. K. A. Baker of Harvard University, who read the manuscript.)

Our proof of the theorem cited in section **3** depends on the possibility of solving equations (3.4) and (3.8), i.e. upon the existence of « additive » inverses. In order to extend the theorem involving the modified distributive law to pseudo-rings, therefore, we must come up with an alternate proof which relies on a cancellation lax for « addition » rather than « additive » inverses.

An alternate proof of this sort proceeds as follows. Since x and y are arbitrary, (2.2) can be translated into

$$a = a^2 + ax + xa \qquad (x \neq 0;\ x \text{ otherwise arbitrary}), \tag{5.1}$$

which in conjuctions with (2.2) as it stands yields

$$ay = ax \qquad (x,\ y \neq 0;\ x,\ y \text{ arbitrary otherwise}). \tag{5.2}$$

(In validating this conclusion we need only resort to the cancellation law for « addition ».) We can set $x = y + y$ in (5.2) and find that

$$ay = a(y + y) = ay + ay\,, \tag{5.3}$$

i.e.

$$ay = 0 \qquad (y \neq 0;\ y \text{ otherwise arbitrary})\,. \tag{5.4}$$

(Since x, y were assumed to be different from « 0 » and $y = x$, $x = y + y \implies x, y = 0$, the substitution is impossible unless our algebraic structure contains three or more elements.) Assuming for the moment that $a \neq 0$ (a was assumed to be arbitrary), we set $y = a$ in (5.4) and $x = a$ in (5.1), winding up with the contradiction

$$a = 0\,. \tag{5.5}$$

References.

[1] N. Jacobson, *Lectures in Abstract Algebra*, Vol. I, D. Van Nostrand, Princeton 1951 (cf. pages 49-52).

Summary.

It is shown that any ring characterized by a dual distributive law («+» is distributive with respect to «·») is necessarily trivial. A modified (weaker) dual distributive law is introduced, and it is shown that any ring satisfying the new law consists of at most two elements. Finally, it is shown that these theorems apply to pseudo-rings, a much wider class of algebraic structures than rings.

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