Riv. Mat. Univ. Parma (2) 6 (1965), 55-64

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On Functions of Bounded ω-Variation. (**)

Let $\omega(x)$ be a function of bounded variation. It is wellknown that the set of the points of discontinuities of $\omega(x)$ is almost enumerable although this set may be everywhere dense [1]. Since $\omega(x)$ can be expressed as the difference of two non-decreasing functions there will be no loss of generality in taking $\omega(x)$ to be non-decreasing. Let $[a, b]$ be a closed interval and suppose that $\omega(x)$ is defined in $[a, b]$ with the understanding that $\omega(x) = \omega(a)$ for $x < a$ and $\omega(x) = \omega(b)$ for $x > b$. Prof. R. L. Jeffery [2] now denotes by $\mathfrak{M}$ the class of functions $F(x)$ defined as follows:

$F(x)$ is defined at points of continuity of $\omega(x)$ on $[a, b]$ and if $\mathcal{S}$ denotes the set over which $\omega(x)$ is continuous, then $F(x)$ is continuous over $\mathcal{S}$ at points of $\mathcal{S}$. At any point of discontinuity x_0 of $\omega(x)$, it is supposed that $F(x)$ tends to a limit as x tends to x_0+ and to x_0- over the points of $\mathcal{S}$. These limits will be denoted by $F(x_0+)$ and $F(x_0-)$. Also for $x < a$, it is assumed that $F(x) = F(a+)$ and for $x > b$, $F(x) = F(b-)$. $F(x)$ may or may not be defined at points of discontinuity of $\omega(x)$.

Prof. Jeffery has introduced the following definition.

Definition. A function $F(x)$ defined on $[a, b]$ and in class $\mathfrak{M}$ is absolutely continuous relative to ω, AC - ω, if for $\varepsilon > 0$ there exists $\delta > 0$ such that for any set of non-overlapping intervals (x_i, x_i') on $[a, b]$ with $\sum\{\omega(x_i'+) - \omega(x_i-)\} < \delta$ the relation $\sum |F(x_i'+) - F(x_i-)| < \varepsilon$ is satisfied.

In [2] some results have been obtained for functions $F(x)$ in class $\mathfrak{M}$ which are AC - ω. In this paper we have introduced the definition of bounded var-

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iation relative to $\omega(x)$ of a function (or BV - ω function) belonging to $\mathfrak{M}$; and obtaining some preliminary results it has been shown that a function $F(x)$ which is AC - ω on $[a, b]$ must be BV - ω on $[a, b]$.

Throughout the paper we shall consider only those functions $F(x)$ in class $\mathfrak{M}$ for which $F(x_0+)$ and $F(x_0-)$, $x_0 \in I - \mathcal{S}$, are finite, where $I = [a, b]$.

Let $\omega(a) = y_0 < y_1 < y_2 < \ldots < y_n = \omega(b)$ be any subdivision of $[\omega(a), \omega(b)]$ where $y_i \in \omega(I)$. For every y_i there is an $x_i \in I$ such that $y_i = \omega(x_i)$. If for an y_i there exist more than one x_i such that $\omega(x_i) = y_i$, we shall take any one x_i. It is obvious that $a \leqslant x_0 < x_1 < x_2 < \ldots < x_n \leqslant b$. We say that the points $x_0, x_1, x_2, \ldots, x_n$ form a subdivision of $[a, b]$ relative to ω or is a ω-subdivision of $[a, b]$. Let $F(x)$ be defined in $[a, b]$ and in class $\mathfrak{M}$ and let

$$V = \sum_{i=1}^{n} | F(x_i+) - F(x_{i-1}-) | .$$

Definition. The least upper bound of the aggregate $\{ V \}$ of sums V for all possible ω-subdivisions of $[a, b]$ is called the total ω-variation of $F(x)$ on $[a, b]$ and is denoted by $V_\omega(F; a, b)$. If $V_\omega(F; a, b) < +\infty$, then $F(x)$ is said to be a function of bounded variation relative to ω, BV - ω, on $[a, b]$.

If $\omega(x)$ is constant in $[\alpha, \beta] \subset [a, b]$, then any function $F(x)$ defined in $[a, b]$ and in class $\mathfrak{M}$ will always be assumed to be BV - ω on $[\alpha, \beta]$.

Theorem 1. *Let $F(x)$ be defined in $[a, b]$ and belong to $\mathfrak{M}$. If $F(x)$ is of bounded variation on $[a, b]$, then it is BV- ω on $[a, b]$.*

Proof. Let D: $(a \leqslant x_0 < x_1 < x_2 < \ldots < x_n \leqslant b)$ be any ω-subdivision of $[a, b]$. We shall show that

$$\sum_{1}^{n} | F(x_i+) - F(x_{i-1}-) | \leqslant 2 \overset{b}{\underset{a}{\mathrm{V}}}(F), \tag{1}$$

where $\overset{b}{\underset{a}{\mathrm{V}}}(F)$ stands for the total variation of $F(x)$ in $[a, b]$. The following four cases come up for consideration

(i) $a < x_0,\ x_n < b$, (ii) $a = x_0,\ x_n < b$,

(iii) $a < x_0,\ x_n = b$, (iv) $a = x_0,\ x_n = b$.

We prove (1) for the case (i). The proof in the other cases will follow similarly.

Choose the points $\xi_0, \xi_1, \xi_2, \ldots, \xi_{n-1}$ and $\eta_1, \eta_2, \ldots, \eta_n$ of $\mathcal{S}$ such that

$$a < \xi_0 < x_0 < \xi_1 < x_1 < \eta_1 < \xi_2 < x_2 < \eta_2 < \ldots < \xi_{n-1} < x_{n-1} < \eta_{n-1} < x_n < \eta_n .$$

Then $[\xi_0, \eta_1]$, $[\xi_2, \eta_3]$, $[\xi_4, \eta_5]$, ... and $[\xi_1, \eta_2]$, $[\xi_3, \eta_4]$, $[\xi_5, \eta_6]$, ... form two sets of non-overlapping intervals in $[a, b]$. Since, by hypothesis, $F(x)$ is of bounded variation in $[a, b]$, we have

$$\sum_i | F(\eta_i) - F(\xi_{i-1}) | \leqslant \overset{b}{\underset{a}{\mathrm{V}}}(F) \qquad (i = 1, 3, 5, ...)$$

and

$$\sum_i | F(\eta_i) - F(\xi_{i-1}) | \leqslant \overset{b}{\underset{a}{\mathrm{V}}}(F) \qquad (i = 2, 4, 6, ...) .$$

So,

$$\sum_{i=1}^{n} | F(\eta_i) - F(\xi_{i-1}) | \leqslant 2 \overset{b}{\underset{a}{V}}(F) .$$

Letting $\xi_i \to x_i -$ and $\eta_i \to x_i +$ over the points of $\mathcal{S}$, we get

$$\sum_{i=1}^{n} | F(x_i+) - F(x_{i-1}-) | \leqslant 2 \overset{b}{\underset{a}{V}}(F) .$$

Since D is any ω-subdivision of $[a, b]$ it follows that $F(x)$ is BV - ω on $[a, b]$. This proves the Theorem.

The following example shows that the converse of the above Theorem is not true. Let $\omega(x)$ and $F(x)$ be defined in $[0, 2]$ as follows:

$$\omega(x) = \begin{cases} 0, & 0 \leqslant x \leqslant 1 \\ x-1 , & 1 < x \leqslant 2 \end{cases}$$

and

$$F(x) = \begin{cases} x \sin \dfrac{\pi}{2x} , & 0 < x \leqslant 2 \\ 0, & x = 0 . \end{cases}$$

Let D: $(0 \leqslant x_0 < x_1 < x_2 < ... < x_n \leqslant 2)$ be any ω-subdivision of $[0, 2]$. Then $0 \leqslant x_0 \leqslant 1$ and $x_1 > 1$.

Now,

$$V = \sum_{i=1}^{n} | F(x_i+) - F(x_{i-1} -) | = \sum_{i=1}^{n} | F(x_i) - F(x_{i-1}) |$$

$$\leqslant | F(x_0) | + | F(x_1) | + \sum_{i=2}^{n} | F(x_i) - F(x_{i-1}) | \leqslant 3 + \overset{2}{\underset{1}{\mathrm{V}}}(F) < M,$$

where M is a fixed constant because $F(x)$ is of bounded variation on $[1, 2]$.

The above inequality is true for any ω-subdivision of $[0, 2]$. Hence $F(x)$ is BV - ω on $[0, 2]$. However, it is well-known that $F(x)$ is not of bounded variation on $[0, 2]$.

Theorem 2. *If $F(x)$ is* BV - ω *on* $[a, b]$, *then* $F(x\pm)$ *is bounded on* $[a, b]$.

Proof. Let $K_1 = V_\omega(F; a, b) + |F(a+)|$, $K_2 = V_\omega(F; a, b) + |F(b-)|$ and $K = \max(K_1, K_2)$.

We consider the following cases.

(i) $\omega(x)$ *is constant in* $[a, b]$.

In this case, since $\omega(x)$ is continuous in $[a, b]$, $F(x)$ is also continuous in $[a, b]$. Consequently $F(x)$ is bounded in $[a, b]$.

(ii) $\omega(a) < \omega(x) < \omega(b)$ *for* $x \in (a, b)$.

In this case the points $a < x < b$ form a ω-subdivision of $[a, b]$. So

$$|F(x+) - F(a-)| + |F(b+) - F(x-)| \leqslant V_\omega(F; a, b).$$

Hence

$$|F(x+)| \leqslant V_\omega(F; a, b) + |F(a+)| = K_1$$

and

$$|F(x-)| \leqslant V_\omega(F; a, b) + |F(b-)| = K_2.$$

So,

$$|F(x\pm)| \leqslant K \quad \text{in } [a, b].$$

(iii) $\omega(x)$ *does not satisfy* (i) *and* (ii).

Let $\omega(x) = \omega(a)$ in $[a, \alpha]$ where $\alpha \geqslant a$ is the upper bound of the set $\{x: x \in [a, b], \omega(x) = \omega(a)\}$ and the interval $[a, \alpha]$ is closed or open in the right according as $\omega(\alpha) = \omega(a)$ or not. Similarly let $\omega(x) = \omega(b)$ in $[\beta, b]$, where $\beta \leqslant b$ is the lower bound of the set $\{x: x \in [a, b], \omega(x) = \omega(b)\}$ and the interval $[\beta, b]$ is closed or open in the left according as $\omega(\beta) = \omega(b)$ or not.

Let K' be the maximum of K, $|F(\alpha+)|$, $|F(\alpha-)|$, $|F(\beta+)|$, $|F(\beta-)|$.

Let $\alpha > a$ *and* $\beta = b$. If $x \in (a, \alpha)$, then $F(x)$ is continuous at x and x, b form a ω-subdivision of $[a, b]$.

So, $|F(b+) - F(x-)| \leqslant V_\omega(F; a, b)$, i.e.

$$|F(x)| \leqslant K_2 \leqslant K \leqslant K'.$$

If $x \in (\alpha, \beta)$, then a, x, b form a ω-subdivision of $[a, b]$. Proceeding as in case (ii), it can be shown that

$$|F(x\pm)| \leqslant K \leqslant K'.$$

So,

$$|F(x\pm)| \leqslant K' \quad \text{in } [a, b].$$

Similarly, considering the cases $\alpha = a$, $\beta < b$ and $\alpha > a$, $\beta < b$ it can be shown that $|F(x\pm)| \leqslant K'$, $x \in [a, b]$. This proves the theorem.

Theorem 3. *If* $F(x)$ *is* BV - ω *on* $[a, c]$ *and* $[c, b]$, *where* $a < c < b$, *then it is* BV - ω *on* $[a, b]$.

Proof. We may suppose that $\omega(x)$ is not constant in $[a, b]$, because in that case $F(x)$ is, by definition, BV - ω on $[a, b]$. Let $a \leqslant x_0 < x_1 < x_2 < \dots < x_n \leqslant b$ be any ω-subdivision of $[a, b]$. If $x_n \leqslant c$, then $a \leqslant x_0 < x_1 < x_2 < \dots < x_n \leqslant c$ forms a ω-subdivision of $[a, c]$ and so

$$V = \sum_{i=1}^{n} |F(x_i+) - F(x_{i-1}-)| \leqslant V_\omega(F; a, c). \tag{2}$$

If $x_n > c$, then for some positive integer $m (\leqslant n)$ $x_{m-1} \leqslant c < x_m$. We consider the following cases.

(i) $\omega(x') < \omega(c) < \omega(x'')$ for $x' \in [a, c)$ and $x'' \in (c, b]$.

(ii) $\omega(x)$ is constant in (α, β), where $a \leqslant \alpha \leqslant c \leqslant \beta \leqslant b$, the two equalities at the ends do not hold simultaneously and the nature of the interval (α, β) is determined analogously to the case (iii) of Theorem 2.

Case (i). If $x_{m-1} = c$, then $x_0, x_1, \dots, x_{m-1}$ and $x_{m-1}, x_m, \dots, x_n$ form ω-subdivisions of $[a, c]$ and $[c, b]$ respectively.

So,

$$V = \sum_{i=1}^{n} |F(x_i+) - F(x_{i-1}-)| = \sum_{1}^{m-1} \dots + \sum_{m}^{n} \dots \tag{3}$$
$$\leqslant V_\omega(F; a, c) + V_\omega(F; c, b).$$

If $x_{m-1} < c$, the points $x_0, x_1, \ldots, x_{m-1}, c$ and $c, x_m, \ldots, x_n$ form ω-subdivisions of $[a, c]$ and $[c, b]$ respectively.

So,

$$(4)\qquad \begin{aligned} V &= \sum_{i=1}^{n} | F(x_i+) - F(x_{i-1}-) | \\ &= \sum_{1}^{m-1} \cdots + | F(x_m +) - F(x_{m-1} -) | + \sum_{m+1}^{n} \cdots \\ &\leqslant \{ \sum_{1}^{m-1} \cdots + | F(c+) - F(x_{m-1} -) | \} + | F(c+) - F(c-) | \\ &\qquad + \{ | F(x_m+) - F(c-) | + \sum_{m+1}^{n} \cdots \} \\ &\leqslant V_\omega(F;\ a,\ c) + V_\omega(F;\ c,\ b) + | F(c+) - F(c-) | . \end{aligned}$$

Case (ii). *Let* $x_{m-1} \bar{\in} (\alpha, \beta)$. If $x_m \bar{\in} (\alpha, \beta)$, then (4) holds. Let $x_m \in (\alpha, \beta)$.

If $m = n$ (consequently $\beta = b$) then $x_0, x_1, \ldots, x_{n-1}, c$ form a ω-subdivision of $[a, c]$ and so

$$(5)\qquad \begin{aligned} V &= \sum_{1}^{n} | F(x_i+) - F(x_{i-1}-) | = \sum_{1}^{n-1} \cdots + | F(x_n+) - F(x_{n-1}-) | \\ &\leqslant \{ \sum_{1}^{n-1} \cdots + | F(c+) - F(x_{n-1} -) | \} + | F(c+) - F(x_n+) | \\ &\leqslant V_\omega(F;\ a,\ c) + | F(c+) | + | F(x_n +) | . \end{aligned}$$

If $m < n$, then $x_0, x_1, \ldots, x_{m-1}, c$ and $x_m, x_{m+1}, \ldots, x_n$ form ω-subdivision of $[a, c]$ and $[c, b]$ respectively.

So,

$$(6)\qquad \begin{aligned} V &= \sum_{1}^{n} | F(x_i+) - F(x_{i-1}-) | = \sum_{1}^{m-1} \cdots + | F(x_m+) - F(x_{m-1}-) | + \sum_{m+1}^{n} \cdots \\ &\leqslant \{ \sum_{1}^{m-1} \cdots + | F(c+) - F(x_{m-1} -) | \} + | F(c+) - F(x_m+) | + \sum_{m+1}^{n} \cdots \\ &\leqslant V_\omega(F;\ a,\ c) + V_\omega(F;\ c,\ b) + | F(c+) | + | F(x_m+) | . \end{aligned}$$

Let $x_{m-1} \in (\alpha, \beta)$. If $x_{m-1} = c$ and $m > 1$, then (3) holds. If $x_{m-1} = c$ and $m = 1$ (consequently $\alpha = a$), then $x_0, x_1, \ldots, x_n$ form a ω-subdivision of $[c, b]$ and

$$(7)\qquad V = \sum_{1}^{n} | F(x_i+) - F(x_{i-1}-) | \leqslant V_\omega(F;\ c,\ b) .$$

If $x_{m-1} < c$, then considering the cases $m = 1$ and $m > 1$ it can be shown that

$$V = \sum_{1}^{n} | F(x_i +) - F(x_{i-1} -) | \tag{8}$$

$$\leqslant V_\omega(F;\ c,\ b) + | F(c-) | + | F(x_0 -) |$$

and

$$V \leqslant V_\omega(F;\ a,\ c) + V_\omega(F;\ c,\ b) + | F(c-) | + | F(x_{m-1} -) | \tag{9}$$

according as $m = 1$ and $m > 1$.

By Theorem 2, $F(x\pm)$ is bounded on $[a, c]$ and $[c, b]$, so there exists a constant K such that

$$| F(x\pm) | \leqslant K \qquad \text{for } x \text{ in } [a,\ b]\,.$$

From (2), (3), ..., (9) it follows that

$$V = V_\omega(F;\ a,\ c) + V_\omega(F;\ c,\ b) + 2K\,. \tag{10}$$

Since (10) holds for any ω-subdivision of $[a, b]$ it follows that $F(x)$ is BV - ω on $[a, b]$. This proves the Theorem.

Theorem 4. *If $F(x)$ is AC - ω on $[a, b]$ and if $\omega(x)$ is constant in $(\alpha, \beta) \subset [a, b]$, then $F(x)$ is also constant in (α, β).*

Proof. Since $\omega(x)$ is continuous in (α, β), $F(x)$ is continuous in (α, β). Choose $\varepsilon > 0$ arbitrary. There exists a $\delta > 0$ such that for any set of non-overlapping intervals (x_i, x_i') in $[a, b]$ with $\sum \{ \omega(x_i' +) - \omega(x_i -) \} < \delta$, we have $\sum | F(x_i' +) - F(x_i -) | < \varepsilon$.

Let $c = \frac{1}{2}(\alpha + \beta)$ and let $x' \in (\alpha, c)$, $x'' \in (c, \beta)$. The intervals (x', c) and (c, x'') are non-overlapping and since $\omega(x)$ is constant in (α, β), we have

$$\{ \omega(c) - \omega(x') \} + \{ \omega(x'') - \omega(c) \} = 0 < \delta\,.$$

So, $| F(c) - F(x') | + | F(x'') - F(c) | < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies that $F(x') = F(c) = F(x'')$ and this proves the Theorem.

Theorem 5. *If $F(x)$ is AC - ω on $[a, b]$, then it is BV - ω on $[a, b]$.*

Proof. Since $F(x)$ is AC - ω on $[a, b]$, there exists $\delta > 0$ such that for any set of non-overlapping intervals (x_i, x_i') on $[a, b]$ for which $\sum_i \{ \omega(x_i'+) - \omega(x_i-) \} < \delta$, we have $\sum_i | F(x_i'+) - F(x_i -) | < 1$.

We consider the following cases.

(I) *The saltus of $\omega(x)$ at every point of $[a, b]$ is less than $\frac{1}{2}\delta$.*

In this case $[a, b]$ can be broken up into a finite number of sub-intervals $[c_0, c_1], [c_1, c_2], \ldots, [c_{N-1}, c_N]$ $(a = c_0 < c_1 < c_2 < \ldots < c_N = b)$ such that

$$\{ \omega(c_r+) - \omega(c_{r-1}-) \} < \tfrac{1}{2}\delta \qquad (r = 1, 2, \ldots, N). \tag{11}$$

Let $c_{r-1} \leqslant x_0 < x_1 < x_2 < \ldots < x_n \leqslant c_r$ be any ω-subdivision of $[c_{r-1}, c_r]$, $1 \leqslant r \leqslant N$. The set of intervals (x_{i-1}, x_i) are non-overlapping and hence by (11)

$$\sum_{i=1}^{n} \{ \omega(x_i+) - \omega(x_{i-1}-) \} < \delta .$$

So,

$$\sum_{i=1}^{n} | F(x_i+) - F(x_{i-1}-) | < 1 .$$

Since this is true for any ω-subdivision of $[c_{r-1}, c_r]$, we have

$$V_\omega(F; c_{r-1}, c_r) \leqslant 1 .$$

Thus $F(x)$ is BV - ω on each of the intervals $[c_0, c_1], [c_1, c_2], \ldots, [c_{N-1}, c_N]$ and consequently by Theorem 3, $F(x)$ is BV - ω on $[a, b]$.

(II) *There exist points in $[a, b]$ at which the saltus of $\omega(x)$ is $\geqslant \frac{1}{2}\delta$.*

It is well-known [3] that these points are finite in number. Let them be $c_1, c_2, c_3, \ldots, c_m$ such that $c_1 < c_2 < \ldots < c_m$. In $[c_{r-1}, c_r]$ we choose points α, β $(\alpha < \beta)$ of $\mathfrak{S}$ such that

$$\omega(\alpha) - \omega(c_{r-1}+) < \tfrac{1}{2}\delta \quad \text{and} \quad \omega(c_r-) - \omega(\beta) < \tfrac{1}{2}\delta . \tag{12}$$

At each point in $[\alpha, \beta]$ the saltus of $\omega(x)$ is less than $\frac{1}{2}\delta$ and so by case (I), $F(x)$ is BV - ω on $[\alpha, \beta]$.

Now, let $c_{r-1} \leqslant x_0 < x_1 < x_2 < \dots < x_n \leqslant \alpha$ be any ω-subdivision of $[c_{r-1}, \alpha]$. If $c_{r-1} < x_0$, then by (12)

$$\sum_{i=1}^{n} \{ \omega(x_i+) - \omega(x_{i-1}-) \} < \delta$$

and so

$$\sum_{i=1}^{n} | F(x_i+) - F(x_{i-1}-) | < 1 .$$

If $c_{r-1} = x_0$, then we can choose a point ξ in $(x_0, x_1) \cap \mathfrak{S}$ such that $| F(x_0+) - F(\xi) | < 1$. Then

$$\sum_{i=1}^{n} | F(x_i+) - F(x_{i-1}-) | \leqslant | F(x_0+) - F(\xi) | + | F(x_0+) - F(x_0-) |$$

$$+ \{ | F(x_1+) - F(\xi) | + \sum_{i=2}^{n} | F(x_i+) - F(x_{i-1}-) | \}$$

$$< 2 + K, \qquad \text{where} \qquad K = | F(c_{r-1}+) - F(c_{r-1}-) | ,$$

since $(\xi, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n)$ is a system of non-overlapping intervals in $[c_{r-1}, \alpha]$ with

$$\{ \omega(x_1 +) - \omega(\xi) \} + \sum_{i=2}^{n} \{ \omega(x_i+) - \omega(x_{i-1}-) \} < \delta .$$

So, in any case

$$\sum_{i=1}^{n} | F(x_i +) - F(x_{i-1}-) | < 2 + K . \tag{13}$$

Since (13) is true for any ω-subdivision of $[c_{r-1}, \alpha]$, it follows that $F(x)$ is BV - ω on $[c_{r-1}, \alpha]$. Similarly, it can be shown that $F(x)$ is BV - ω on $[\beta, c_r]$. So, by Theorem 3, $F(x)$ is BV - ω on $[c_{r-1}, c_r]$ and consequently by the same Theorem, BV - ω on $[a, b]$. This proves the Theorem.

Finally, the author is thankful to Dr. B. K. Lahiri for his kind help and suggestions in the preparation of the paper.

References.

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