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R. P. GOYAL (*)

Convergence of the Generalized Whittaker Transform. (**)

0. - Introduction.

Several generalizations of the LAPLACE integral

$$f(s) = \int_0^\infty e^{-st}\, a(t)\, \mathrm{d}t \tag{0.1}$$

or the LAPLACE-STIELTJES integral

$$f(s) = \int_0^\infty e^{-st}\, \mathrm{d}\alpha(t) \tag{0.2}$$

have been given by MEIJER [2], BOAS [1] and VARMA [4], [5]. One of the generalisations given by the last author is in the form

$$f(s) = s\int_0^\infty (2st)^{-1/4}\, W_{k,m}(2st)\, a(t)\, \mathrm{d}t\,. \tag{0.3}$$

We have taken our transform in the form

$$f(s) = \int_0^\infty (2st)^{\lambda}\, W_{k,m}(2st)\, a(t)\, \mathrm{d}t, \tag{0.4}$$

(*) Indirizzo: Balwant Rajput College, Agra, India.
(**) Ricevuto: 22-IX-1964.

where $W_{k,m}(x)$ denotes WHITTAKER function. We shall call $f(s)$ as the Generalised WHITTAKER Transform of $a(t)$. When $\lambda = -1/4$, $k = 1/4$ and $m = \pm 1/4$, this reduces to (0.1) due to identity

$$(2st)^{-1/4}\, W_{1/4,\pm 1/4}(2st) \equiv e^{-st}.$$

In this paper I have given the formulae for the abscissa of convergence and uniform convergence for the transform (0.4).

1. – In the following theorem we shall establish the relation between the order properties of the determining function and the convergence properties of the corresponding Generalised WHITTAKER Transform.

We shall need the following result:

Lemma. *If $\alpha(t)$ be a normalized function of bounded variation and the integrals involved exist, then*

$$(1.1)\quad \begin{cases} \displaystyle\int_0^\infty (2st)^\lambda\, W_{k,m}(2st)\, \mathrm{d}\alpha(t) = \\ \displaystyle = s\int_0^\infty (2st)^\lambda\, W_{k,m}(2st)\, \alpha(t)\, \mathrm{d}t - 2s(k+\lambda)\int_0^\infty (2st)^{\lambda-1}\, W_{k,m}(2st)\, \alpha(t)\, \mathrm{d}t + \\ \displaystyle + 2s\left\{ m^2 - \big(k-(1/2)\big)^2 \right\}\int_0^\infty (2st)^{\lambda-1}\, W_{k-1,m}(2st)\, \alpha(t)\, \mathrm{d}t. \end{cases}$$

Proof. We have

$$\mathrm{d}\left\{ x^\lambda\, W_{k,m}(x) \right\} = x^{\lambda-1}\left\{ \lambda\, W_{k,m}(x) + x\, W'_{k,m}(x) \right\} =$$

$$= x^{\lambda-1}[\lambda\, W_{k,m}(x) + \big(k-(1/2)x\big)W_{k,m}(x) - \left\{ m^2 - \big(k-(1/2)\big)^2 \right\} W_{k-1,m}(x)] =$$

{ by using a known recurrence formula [6] }

$$= x^{\lambda-1}[\big(k+\lambda-(1/2)x\big)W_{k,m}(x) - \left\{ m^2 - \big(k-(1/2)\big)^2 \right\} W_{k-1,m}(x)]\,.$$

Therefore

$$\int_0^\infty (2st)^\lambda\, W_{k,m}(2st)\, \mathrm{d}\alpha(t) = (2st)^\lambda\, W_{k,m}(2st)\, \alpha(t)\, \Big|_0^\infty -$$

$$- 2s\int_0^\infty (2st)^{\lambda-1}[(k+\lambda-st)W_{k,m}(2st) - \left\{ m^2 - \big(k-(1/2)\big)^2 \right\} W_{k-1,m}(2st)]\alpha(t)\, \mathrm{d}t\,.$$

But, since $\alpha(t)$ is a normalized function of bounded variation, the integrated portion evidently vanishes at $t=0$. Also for large values of t, $W_{k,m}(t) \sim \sim e^{-(1/2)t}\, t^k$, hence it vanishes also as $t \to \infty$.

Thus result (1.1) is established.

Theorem 1. *If*

$$\alpha(t) = O(e^{\gamma t}\, t^n) \qquad (t \to \infty)$$

for some real numbers γ and n, then the integral

$$\int_0^\infty (2st)^\lambda\, W_{k,m}(2st)\, d\alpha(t) \tag{1.2}$$

converges for $\sigma > \gamma$, where σ is the real part of s, $\alpha(t)$ being a normalized function of bounded variation, provided $\mathrm{Re}\big(\lambda + n \pm m + (1/2)\big) > 0$.

Proof. The hypothesis implies the existence of a constant M such that

$$|\,\alpha(t)\,| \leqslant M\, e^{\gamma t}\, t^n \qquad (0 \leqslant t < \infty),$$

since we have assumed $\alpha(t)$ to be a function of bounded variation. Hence

$$s \int_0^\infty (2st)^\lambda\, W_{k,m}(2st)\, \alpha(t)\, dt < \frac{M}{2\,|\,2s\,|^n} \int_0^\infty (2st)^{\lambda+n}\, e^{\{\gamma/(2s)\}2st}\, W_{k,m}(2st)\, d(2st)\,.$$

Now using Slater's [3] result

$$\int_0^\infty e^{-st}\, t^{\lambda-1}\, W_{k,m}(t)\, dt = \frac{\Gamma(\lambda+m+(1/2))\,\Gamma(\lambda-m+(1/2))}{\Gamma(\lambda-k+1)}\,\cdot \tag{1.3}$$

$$\cdot\, {}_2F_1\big(\lambda \pm m + (1/2);\ \lambda - k + 1;\ (1/2) - s\big)\ (^1)\,,$$

$$\mathrm{Re}\,\big(s + (1/2)\big) > 0 \quad \text{and} \quad \mathrm{Re}\,\big(\lambda \pm m + (1/2)\big) > 0\,,$$

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(1) Here ${}_2F_1\big(\lambda \pm m + (1/2);\ \lambda - k + 1;\ (1/2) - s\big)$ stands for ${}_2F_1\big(\lambda + m + (1/2),\ \lambda - m + (1/2);\ \lambda - k + 1;\ (1/2) - s\big)$.

we have

$$(1.4)\quad \begin{cases} s\int_0^\infty (2st)^\lambda W_{k,m}(2st)\,\alpha(t)\,dt < \\ < \dfrac{M}{2\,|\,2s\,|^n}\,\dfrac{\Gamma(\lambda+n+m+(3/2))\Gamma(\lambda+n-m+(32/))}{\Gamma(\lambda+n-k+2)}\,\cdot \\ \cdot{}_2F_1\big(\lambda+n\pm m+(3/2)\,;\,\lambda+n-k+2;\ (1/2)+\{\gamma/(2s)\}\big) \end{cases}$$

provided $\mathrm{Re}\big(\lambda+n\pm m+(3/2)\big)>0$ and $\mathrm{Re}\big(-\{\gamma/(2s)\}+(1/2)\big)>0$.
Similarly

$$(1.5)\quad \begin{cases} 2s\int_0^\infty (2st)^{\lambda-1} W_{k,m}(2st)\,\alpha(t)\,dt < \\ < \dfrac{M}{|\,2s\,|^n}\,\dfrac{\Gamma(\lambda+n+m+(1/2))\Gamma(\lambda+n-m+(1/2))}{\Gamma(\lambda+n-k+1)}\,\cdot \\ \cdot{}_2F_1\big(\lambda+n\pm m+(1/2)\,;\,\lambda+n-k+1;\ (1/2)+\{\gamma/(2s)\}\big), \end{cases}$$

provided $\mathrm{Re}\big(\lambda+n\pm m+(1/2)\big)>0$ and $\mathrm{Re}\big(-\{\gamma/(2s)\}+(1/2)\big)>0$, and

$$(1.6)\quad \begin{cases} 2s\int_0^\infty (2st)^{\lambda-1} W_{k-1,m}(2st)\,\alpha(t)\,dt < \\ < \dfrac{M}{|\,2s\,|^n}\,\dfrac{\Gamma(\lambda+n+m+(1/2))\,\Gamma(\lambda+n-m+(1/2))}{\Gamma(\lambda+n-k+2)}\,\cdot \\ \cdot{}_2F_1\big(\lambda+n\pm m+(1/2)\,;\,\lambda+n-k+2;\ (1/2)+\{\gamma/(2s)\}\big), \end{cases}$$

provided $\mathrm{Re}\,(\lambda+n\pm m+(1/2))>0$ and $\mathrm{Re}\,(-\{\gamma/(2s)\}+(1/2))>0$.

The series on the right hand side of the inequalities (1.4), (1.5) and (1.6) are convergent if $\mathrm{Re}\,(\{\gamma/(2s)\}+(1/2))<1$, i.e., if $\gamma<\mathrm{Re}\,s=\sigma$ and thus the integrals to the left hand side of these inequalities converge absolutely for $\sigma>\gamma$, provided $\mathrm{Re}\,(\lambda+n\pm m+(1/2))>0$.

Therefore by the Lemma, the integral (1.2) converges for $\sigma>\gamma$ provided $\mathrm{Re}\,(\lambda+n\pm m+(1/2))>0$.

Hence our theorem is established.

Corollary. *If $\alpha(\infty)$ exists and if*

$$\alpha(t)-\alpha(\infty)=O(e^{\gamma t}\,t^n) \qquad (t\to\infty)$$

for real numbers γ and n, then the integral (1.2) converges for $\sigma>\gamma$.

This follows immediately, for

$$\int_0^\infty (2st)^\lambda W_{k,m}(2st)\, \mathrm{d}\{\alpha(t)-\alpha(\infty)\} = \int_0^\infty (2st)^\lambda W_{k,m}(2st)\, \mathrm{d}\alpha(t).$$

2. – We now proceed from convergence properties of the integral to the order properties of $\alpha(t)$.

Theorem 2(a). *If*

$$\int_0^\infty (2st)^\lambda W_{k,m}(2st)\, \mathrm{d}\alpha(t) \tag{2.1}$$

converges for $s = s_0 = \gamma + i\delta$ *with* $\gamma > 0$, *then*

$$\alpha(t) = o(e^{\gamma t}\, t^{-\lambda-k}) \qquad (t \to \infty).$$

Proof. We have

$$\alpha(t)-\alpha(0) = \int_0^t \mathrm{d}\alpha(u) = \int_0^t \frac{\mathrm{d}\beta(u)}{(2s_0u)^\lambda W_{k,m}(2s_0u)},$$

where

$$\beta(t) = \int_0^t (2s_0u)^\lambda W_{k,m}(2s_0u)\, \mathrm{d}\alpha(u) \qquad (0<t<\infty).$$

Integration by parts gives

$$\alpha(t)-\alpha(0) = \beta(t)\frac{1}{(2s_0t)^\lambda W_{k,m}(2s_0t)} - \int_0^t \frac{\mathrm{d}}{\mathrm{d}u}\{(2s_0u)^\lambda W_{k,m}(2s_0u)\}^{-1}\beta(u)\,\mathrm{d}u,$$

since $\beta(0)=0$. By hypothesis $\beta(\infty)$ exists and therefore

$$\lim_{t\to\infty}\{[\alpha(t)-\alpha(0)](2s_0t)^\lambda W_{k,m}(2s_0t)\} =$$

$$= \beta(\infty) - \lim_{t\to\infty}\left((2s_0t)^\lambda W_{k,m}(2s_0t)\int_0^t \frac{\mathrm{d}}{\mathrm{d}u}\{(2s_0u)^\lambda W_{k,m}(2s_0u)\}^{-1}\beta(u)\,\mathrm{d}u\right) =$$

$$= \lim_{t\to\infty}\left((2s_0t)^\lambda W_{k,m}(2s_0t)\int_0^t \frac{\mathrm{d}}{\mathrm{d}u}\{2s_0u)^\lambda W_{k,m}(2s_0u)\}^{-1}[\beta(\infty)-\beta(u)]\,\mathrm{d}u\right) =$$

$$= \lim_{t\to\infty}\frac{\displaystyle\int_0^t \frac{\mathrm{d}}{\mathrm{d}u}\{(2s_0u)^\lambda W_{k,m}(2s_0u)\}^{-1}[\beta(\infty)-\beta(u)]\,\mathrm{d}u}{\{(2s_0t)^\lambda W_{k,m}(2s_0t)\}^{-1}},$$

which is easily seen to be zero if the real part of s_0 is positive.

Also $W_{k,m}(x) \sim e^{-(1/2)x} x^k \quad (x \to \infty)$. Therefore

$$\lim_{t\to\infty} \left\{ [\alpha(t) - \alpha(0)]\, (2s_0 t)^{2+k}\, e^{-s_0 t} \right\} = 0 .$$

Hence $\alpha(t) - \alpha(0) = o(e^{\gamma t} t^{-\lambda-k}) \quad (t \to \infty)$, or $\alpha(t) = o(e^{\gamma t}\, t^{-\lambda-k}) \quad (t \to \infty)$.

Thus the theorem is proved.

If we put $\lambda = -1/4$, $k = 1/4$, we get $\alpha(t) = o(e^{\gamma t})$, which is a known result in the theory of ordinary LAPLACE transform.

Theorem 2(b). *If the integral* (2.1) *converges for* $s = s_0 = \gamma + i\delta$ *with* $\gamma < 0$ *and if* $\alpha(\infty)$ *exists, then*

$$\alpha(t) - \alpha(\infty) = o(e^{\gamma t}\, t^{-\lambda-k}) \qquad (t \to \infty).$$

Proof. We have

$$\alpha(\infty) - \alpha(t) = \int_t^\infty d\alpha(u) = \int_t^\infty \left\{ (2s_0 u)^\lambda\, W_{k,m}(2s_0 u) \right\}^{-1} d\beta(u) ,$$

where

$$\beta(t) = \int_0^t (2s_0 u)^\lambda\, W_{k,m}(2s_0 u)\, d\alpha(u).$$

Integration by parts gives

$$\alpha(\infty) - \alpha(t) = -\left\{ (2s_0 t)^\lambda W_{k,m}(2s_0 t) \right\}^{-1} \beta(t) - \int_t^\infty \frac{d}{du} \left\{ (2s_0 u)^\lambda\, W_{k,m}(2s_0 u) \right\}^{-1} \beta(u)\, du .$$

By hypothesis $\beta(\infty)$ exists, therefore

$$\lim_{t\to\infty} \left\{ [\alpha(\infty) - \alpha(t)](2s_0 t)^\lambda\, W_{k,m}(2s_0 t) \right\} =$$

$$= -\beta(\infty) - \lim_{t\to\infty} \left\{ (2s_0 t)^\lambda\, W_{k,m}(2s_0 t) \int_t^\infty \frac{d}{du}\, [(2s_0 u)^\lambda\, W_{k,m}(2s_0 u)]^{-1}\, \beta(u)\, du \right\} =$$

$$= \lim_{t\to\infty} \left\{ (2s_0 t)^\lambda\, W_{k,m}(2s_0 t) \int_t^\infty \frac{d}{du}\, [(2s_0 u)^\lambda\, W_{k,m}(2s_0 u)]^{-1}\, [\beta(\infty) - \beta(u)]\, du \right\} .$$

The last limit can be proved equal to zero as in Theorem 2(a). Therefore

$$\lim_{t\to\infty}\left\{ [\alpha(\infty)-\alpha(t)]e^{-s_0 t}\,(2s_0 t)^{\lambda+k}\right\}=0\ ,$$

whence

$$\alpha(t)-\alpha(\infty)=o(e^{\gamma t}\,t^{-\lambda-k}) \qquad (t\to\infty).$$

Thus the theorem is proved.

The corresponding result in the theory of Ordinary LAPLACE Transform is again obtained by taking $\lambda=-1/4$ and $k=1/4$.

3. – We shall now establish the formula for the abscissa of convergence. We shall take $n=-(\lambda+k)$ throughout this section.

Theorem 3(a). *If*

$$\lim_{t\to\infty}\left\{ (1/t)\log|\,\alpha(t)/t^n\,|\right\}=l\neq 0,$$

then l is the abscissa of convergence σ_c for the integral

$$\int_0^\infty (2st)^\lambda\ W_{k,m}(2st)\ d\alpha(t). \tag{3.1}$$

Proof. Case I: $l>0$.

We shall first prove that (3.1) converges for $\sigma>l$.

Let ε be any arbitrary positive constant, then by hypothesis

$$\alpha(t)/t^n=O(e^{(l+\varepsilon)t})$$

or

$$\alpha(t)=O(e^{(l+\varepsilon)t}\ t^n). \tag{3.2}$$

Hence by Theorem 1, the integral (3.1) converges for $\sigma>l+\varepsilon$, that is it converges for $\sigma>l$.

Let us now prove that (3.1) diverges for $\sigma<l$.

Suppose it converges for $s=\gamma$, where $0<\gamma<l$, then by Theorem 2(a), we should have

$$\alpha(t)=o(e^{\gamma t}\,t^n) \qquad (t\to\infty).$$

This implies the existence of constants M and T such that

$$|\alpha(t)| < M e^{\gamma t} t^n \qquad (T < t < \infty),$$

whence

$$\log|\alpha(t)| < \log M + \gamma t + n \log t$$

and so

$$l = \overline{\lim_{t \to \infty}} \{ (1/t) \log|\alpha(t)/t^n| \} < \gamma,$$

which contradicts our assumption that $\gamma < l$.

Thus our theorem is established for positive l.

Case II: $l < 0$.

If $l < 0$, the same argument as used for positive l shows that (3.1) converges for $\sigma > l$.

Now by hypothesis

$$\alpha(t) = t^n e^{(l+\varepsilon)t},$$

where l is negative. Thus the hypothesis implies that $\alpha(\infty) = 0$.

Now to prove that (3.1) diverges for $\sigma < l$, let us suppose that it converges for $\sigma = \gamma < l$, then by Theorem 2(b), we should have

$$\alpha(t) - \alpha(\infty) = o(e^{\gamma t} t^n) \qquad (t \to \infty),$$

whence

$$l = \overline{\lim_{t \to \infty}} \{ (1/t) \log|\alpha(t)/t^n| \} \leqslant \gamma,$$

which is contradictory to the assumption that $\gamma < l$. Hence (3.1) must diverge for $\sigma < l$.

Thus the theorem is fully proved.

Corollary. *If the integral* (3.1) *has a non-negative abscissa of convergence* σ_c, *then it is easy to see from the above theorem that*

$$\sigma_c = \overline{\lim_{t \to \infty}} \{ (1/t) \log|\alpha(t)/t^n| \}.$$

Theorem 3(b). *If*

$$\lim_{t\to\infty}\{(1/t)\log|\alpha(t)/t^n|\}=0$$

and if $t^{-n}\alpha(t)$ *approaches no limit as* t *becomes infinite, then* $\sigma_c=0$.

Proof. If ε be an arbitrary positive constant, then

$$\alpha(t)=O(e^{\varepsilon t}t^n) \qquad (t\to\infty)$$

and since $t^{-n}\alpha(t)$ approaches no limit, ε cannot vanish. Hence the integral (3.1) must converge for $\sigma>0$ and diverge for $s=0$.

Thorem 3(c). *If* $\alpha(\infty)$ *exists and if*

$$\overline{\lim_{t\to\infty}}\{(1/t)\log|(\alpha(t)-\alpha(\infty))/t^n|\}=l_1\leqslant 0, \tag{3.3}$$

then $\sigma_c=l_1$.

Proof. If we take $l=0$ in (3.2), then we get

$$\alpha(t)=O(e^{\varepsilon t}t^n) \qquad (t\to\infty),$$

where ε is any positive constant, and so

$$\alpha(t)-\alpha(\infty)=O(e^{\varepsilon t}t^n) \qquad (t\to\infty).$$

Thus it is clear that for $l=0$ (3.3) holds whenever $\alpha(\infty)$ exists.

Now the corollary of Theorem 1 shows that (3.1) converges for $\sigma>l_1$. On the other hand if (3.1) converged for $\sigma=\gamma<l_1$, then γ would be negative and by Theorem 2(b) we must have

$$\alpha(\infty)-\alpha(t)=o(e^{\gamma t}t^n) \qquad (t\to\infty),$$

whence

$$l_1=\overline{\lim_{t\to\infty}}\{(1/t)\log|(\alpha(\infty)-\alpha(t))/t^n|\}\leqslant\gamma,$$

which contradicts our assumption that $\gamma<l_1$. Thus the integral (3.1) diverges for $\sigma<l_1$.

Hence our theorem is proved.

Corollary. *If* (3.1) *has a negative abscissa of convergence* σ_c *and* $\alpha(\infty)$ *exists, then*

$$\sigma_c=\lim_{t\to\infty}\{(1/t)\log|(\alpha(\infty)-\alpha(t))/t^n\}.$$

The proof follows from the above theorem.

4. - Uniform convergence near the axis of convergence.

It is a common property of LAPLACE integrals that they converge uniformly in an angular sector near the abscissa of convergence. We shall show here that the integral

$$f(s)=\int_0^\infty (2st)^\lambda\, W_{k,m}(2st)\,\mathrm{d}\alpha(t) \tag{4.1}$$

also has a similar property.

Since $W_{k,m}(2st)\sim e^{-st}(2st)^k$ as $t\to\infty$, the following Lemma is evident.

Lemma. *If t is large and s and s_0 are non-zero, then*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{(2st)^\lambda\, W_{k,m}(2st)}{(2s_0t)^\lambda\, W_{k,m}(2s_0t)}\right|\leqslant |s/s_0|^{\lambda+k}\,|s-s_0|\,e^{-(\sigma-\sigma_0)t}. \tag{4.2}$$

Theorem 4. *If the integral (4.1) converges at $s=s_0=\sigma_0+i\tau_0$, and if H and K are any constants such that $H>0$, $K>1$, then the integral (4.1) converges uniformly in the region Δ defined by the inequality*

$$|s/s_0|^{\lambda+k}\,|s-s_0|\leqslant K\,(\sigma-\sigma_0)\,e^{H(\sigma-\sigma_0)}. \tag{4.3}$$

Proof. We observe that if s lies in Δ we must have $\sigma>\sigma_0$ or else $s=s_0$. Hence, since (4.1) converges at $s=s_0$, it must converge in Δ.

If ε be an arbitrary positive number we have to prove that we can determine a number R_0 independent of s in Δ such that, for $R>R_0$,

$$\left|\int_R^\infty (2st)^\lambda\, W_{k,m}(2st)\,\mathrm{d}\alpha(t)\right|<\varepsilon. \tag{4.4}$$

Let us take

$$\beta(t)=\int_0^t (2s_0u)^\lambda\, W_{k,m}(2s_0u)\,\mathrm{d}\alpha(u) \qquad (0\leqslant t<\infty)$$

and determine R_0 greater than H and such that

$$|\beta(t)-\beta(t')|<\varepsilon/K$$

for all values fo t and t' greater than R_0. This is possible by the convergence of (4.1) at s_0. Thus

$$\int_R^\infty (2st)^\lambda\, W_{k,m}(2st)\,\mathrm{d}\alpha(t)=\int_R^\infty \frac{(2st)^\lambda\, W_{k,m}(2st)}{(2s_0t)^\lambda\, W_{k,m}(2s_0t)}\,\mathrm{d}[\beta(t)-\beta(R)]\,.$$

Hence if $R > R_0$, we finally get with the help of Lemma (4.2) that

$$\left| \int_R^\infty (2st)^\lambda W_{k,m}(2st)\, d\alpha(t) \right| \leqslant (\varepsilon/K) | s/s_0 |^{2+k} \frac{|s - s_0|}{\sigma - \sigma_0} e^{-(\sigma-\sigma_0)R}.$$

Hence if s is any point of Δ other than s_0, then

$$\left| \int_R^\infty (2st)^\lambda W_{k,m}(2st)\, d\alpha(t) \right| \leqslant \varepsilon\, e^{-(\sigma-\sigma_0)(R-H)} < \varepsilon ,$$

and in case $s = s_0$

$$\left| \int_R^\infty (2st)^\lambda W_{k,m}(2st)\, d\alpha(t) \right| = | \beta(\infty) - \beta(R) | \leqslant \varepsilon/K < \varepsilon.$$

Thus the theorem is proved.

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