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On the Absolute Nörlund Summability Factors. (**)

1. - Let s_n denote the n-th partial sum of a given infinite series $\sum a_n$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = \sum_{i=0}^{n} p_i, \qquad P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n} \sum p_{n-r}\, s_r \qquad (P_n \neq 0) \tag{1.1}$$

defines the sequence $\{T_n\}$ of NÖRLUND means of the sequence $\{s_n\}$ generated by the coefficients $\{p_n\}$ [6].

The series $\sum a_n$ is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if the series $\sum |T_n - T_{n-1}|$ is convergent [4]. In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\,\Gamma(\alpha)} \qquad (\alpha > 0), \tag{1.2}$$

the NÖRLUND mean reduces to (C, α) mean [1]. Thus, the summability $|N, p_n|$,

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where p_n is defined by (1.2), is the same as $|C, \alpha|$. Again, when

$$p_n = \frac{1}{n+1}, \tag{1.3}$$

the NÖRLUND mean reduces to the harmonic mean.

The conditions for the regularity of the method of summability (N, p_n) defined by (1.1) are

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0 \tag{1.4}$$

and

$$\sum_{i=0}^{n} |p_i| = O(P_n), \quad \text{as} \quad n \to \infty. \tag{1.5}$$

If p_n is real and non-negative, (1.5) is automatically satisfied, and then (1.4) is the necessary and sufficient condition for the regularity of the method.

2. – The series $\sum a_n$ is said to be absolutely summable (R, $\log n$, 1), or summable $|R, \log n, 1|$, if for

$$R_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{s_k}{k}$$

the infinite series $\sum |R_n - R_{n-1}|$ is convergent.

3. – Given a sequence $\{\lambda_n\}$, if the series $\sum a_n \lambda_n$ is absolutely summable in some sense, while in general $\sum a_n$ is itself not so summable, then $\{\lambda_n\}$ is said to be the absolute summability factors of the series $\sum a_n$.

KOGBETLIANTZ had proved the following theorem [2] on summability factors for absolute CESÀRO summability:

If a series $\sum a_n$ *is* $|C, \alpha|$ *summable, then the series* $\sum a_n \varepsilon_n$ *is summable* $|C, \beta|$ *for* $\beta \leqslant \alpha$, $\alpha, \beta > 0$, *if* $\varepsilon_n = 1/(n+1)^{\alpha-\beta}$.

In 1952 PEYERIMHOFF gave a simpler proof of the above theorem [8].

The object of this paper is to establish a similar theorem for the case of NÖRLUND summability when the series is summable $|C, 1|$.

In what follows we prove the following

Theorem: *If a series* $\sum a_n$ *is* $|C, 1|$ *summable and if* $\{p_n\}$ *be a non-increasing sequence of real and non-negative numbers, then the series* $\sum a_n P_n/n$ *is* $|N, p_n|$ *summable, where* $P_n = \sum_{i=0}^{n} p_i$.

4. - Proof of the Theorem. For the series $\sum a_n P_n/n$, we have the NÖRLUND mean

$$T_n = (1/P_n) \sum_{r=1}^{n} p_{n-r}\, s_r = (1/P_n) \sum_{r=1}^{n} P_{n-r}\, u_r,$$

where $u_r = a_r P_r/r$.

Now, since $P_{-1} = 0$,

$$T_{n+1} - T_n = \sum_{r=1}^{n+1} \left(\frac{P_{n+1-r}}{P_{n+1}} - \frac{P_{n-r}}{P_n}\right) u_r = \sum_{r=1}^{n+1} r\, a_r \left(\frac{P_{n+1-r}}{P_{n+1}} - \frac{P_{n-r}}{P_n}\right) \frac{P_r}{r^2}.$$

Applying ABEL's transformation and denoting $t_r = \sum_{r=1}^{r} r\, a_r$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$, we have

$$T_{n+1} - T_n = \sum_{r=1}^{n} t_r\, \Delta\left\{\left(\frac{P_{n-1-r}}{P_{n+1}} - \frac{P_{n-r}}{P_n}\right)\frac{P_r}{r^2}\right\} + P_0 \frac{t_{n+1}}{(n+1)^2}.$$

Hence:

$$\sum_{n=1}^{m} |T_{n+1} - T_n| \leqslant \sum_{n=1}^{m} \left| \sum_{r=1}^{n} t_r\, \Delta\left\{\left(\frac{P_{n+1-r}}{P_{n+1}} - \frac{P_{n-r}}{P_n}\right)\frac{P_r}{r^2}\right\}\right| + P_0 \sum_{n=1}^{m} \frac{|t_{n+1}|}{(n+1)^2}$$

$$\leqslant \sum_{n=1}^{m} \left| \sum_{r=1}^{n} t_r \left(\frac{P_{n+1-r}}{P_{n+1}} - \frac{P_{n-r}}{P_n}\right) \Delta \frac{P_r}{r^2}\right| +$$

$$+ \sum_{n=1}^{m} \left| \sum_{r=1}^{n} t_r \frac{P_{r+1}}{(r+1)^2}\, \Delta\left(\frac{P_{n+1-r}}{P_{n+1}} - \frac{P_{n-r}}{P_n}\right)\right| + P_0 \sum_{n=1}^{m} \frac{|t_{n+1}|}{(n+1)^2}$$

$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

Since $\{p_n\}$ is a non-negative, non-increasing sequence, it is easy to see that $\frac{P_{n+1-r}}{P_{n-r}} \geqslant \frac{P_{n+1}}{P_n}$ for all $r \leqslant n$, and hence

$$\sum_1 \leqslant \sum_{r=1}^{m} \left| t_r \Delta \frac{P_r}{r^2} \right| \sum_{n=r}^{m} \left(\frac{P_{n+1-r}}{P_{n+1}} - \frac{P_{n-r}}{P_n} \right) = A \sum_{r=1}^{m} \left| t_r \Delta \frac{P_r}{r^2} \right| \frac{P_{m+1-r}}{P_{m+1}}$$

$$\leqslant A \sum_{r=1}^{m} |t_r| \left| -\frac{P_{r+1}}{r^2} + P_{r+1} \Delta \frac{1}{r^2} \right| \leqslant A \sum_{r=1}^{m} \frac{|t_r|}{r^2} + O\left[\sum_{r=1}^{m} \frac{P_{r+1} |t_r|}{r^3} \right],$$

where A is a positive constant not necessarily the same one each time it occurs.

Now, since $\sum a_n$ is $|C, 1|$ summable, $\sum |t_r|/r^2$ is convergent. Then, since $P_{r+1} \leqslant (r+1) p_0$, we have

$$\sum_1 \leqslant A \sum_{r=1}^{m} \frac{|t_r|}{r^2} + O\left[\sum_{r=1}^{m} \frac{|t_r|}{r^2} \right] = O(1), \quad \text{as } m \to \infty .$$

Further

$$\sum_2 = \sum_{n=1}^{m} \left| \sum_{r=1}^{n} t_r \frac{P_{r+1}}{(r+1)^2} \left(\frac{p_{n+1-r}}{P_{n+1}} - \frac{p_{n-r}}{P_n} \right) \right|$$

$$\leqslant \sum_{r=1}^{m} \frac{|t_r| P_{r+1}}{(r+1)^2} \sum_{n=r}^{m} \left(\frac{p_{n-r}}{P_n} - \frac{p_{n+1-r}}{P_{n+1}} \right) = A \sum_{r=1}^{m} \frac{|t_r| P_{r+1} p_0}{(r+1)^2 P_r}$$

$$= O\left[\sum \frac{|t_r|}{r^2} \right] = O(1), \quad \text{as } m \to \infty .$$

Also:

$$\sum_3 = O(1), \quad \text{as } m \to \infty, \text{ from the hypothesis.}$$

Hence $\sum | T_{n+1} - T_n | < \infty$, which proves the Theorem.

5. – Incidently it can be seen that the theorem, coupled with known results, leads to some important corollaries for $|N, p_n|$ summability.

It is known that whenever $\sum a_n$ is $|R, \log n, 1|$ summable, $\sum a_n/\log n$ is summable $|C, 1|$ [5], [10]. Hence we have the following result:

Corollary I. *If a series* $\sum a_n$ *is* $|R, \log n, 1|$ *summable, then the series* $\sum \frac{a_n P_n}{n \log(n+1)}$ *is summable* $|N, p_n|$.

Again, PRASAD and BHATT [9] (see also PATI [7]) have proved that if $\{\lambda_n\}$ be a convex sequence such that $\sum \lambda_n/n$ is convergent, and if t_n denotes the CESÀRO mean of order one of the sequence $\{n\, a_n\}$ and if

$$t_n = O[\{\log(n+1)\}^k] \ (C,\ 1),$$

then the series $\sum a_n \lambda_n \{\log(n+1)\}^{-k}$ is summable $|C,\ 1|$. This result, combined with the theorem, leads to another important result.

Corollary II. *If $\{\lambda_n\}$ be a convex sequence such that $\sum \lambda_n/n$ is convergent and if t_n denotes the* CESÀRO *mean of order one of the sequence $\{n\, a_n\}$ and if*

$$t_n = O[\{\log(n+1)\}]^k \ (C,\ 1)\,,$$

then the series $\sum a_n \lambda_n \{\log(n+1\}^{-k} P_n/n$ is summable $|N,\ p_n|$.

This generalises a recent result of LAL [3].

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$, and let the FOURIER series of $f(t)$ be given by

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t).$$

We know that the convergence of FOURIER series can be ensured by a local hypothesis, that is to say, the behaviour of the convergence of FOURIER series for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, and we also know that $s_n = O(1)$ implies $t_n = O(1)$. A necessary consequence of Cor. II, then, is the following result due to TRIPATHI [11].

Corollary III. *If $\{p_n\}$ is a non-negative, non-increasing sequence of real constants, and $\{\lambda_n\}$ be a convex sequence of numbers such that $\sum n^{-1}\lambda_n$ is convergent, then $|N,\ p_n|$ summability of $\sum A_n(t)\, \lambda_n \dfrac{P_n}{n}$ can be ensured by a local hypothesis.*

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Summary.

The paper is devoted to the study of absolute Nörlund summability of the series $\sum a_n \mu_n$ *when the series* $\sum a_n$ *is summable* $|C, 1|$.

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