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Some Theorems on Generalized Laplace Transform. - III (**)

1. - Introduction.

A generalization of the LAPLACE transform

$$\Phi(p) = p \int_0^\infty e^{-pt} \, h(t) \, dt, \tag{1.1}$$

in the form of the integral equation

$$\Phi(p) = p \int_0^\infty e^{-\frac{1}{2}pt} \, (pt)^{m-\frac{1}{2}} \, W_{k,m}(pt) \, h(t) \, dt, \tag{1.2}$$

where $W_{k,m}(t)$ denotes the WHITTAKER's function, was introduced, about a decade earlier, by R. S. VARMA ([7], p. 209).

We define $\Phi(p)$ as the operational image of $h(t)$ and $h(t)$ its original in the generalized LAPLACE transform defined by (1.2).

Recently in my previous papers [4] and [5], some theorems for this generalized LAPLACE transform were proved and the images of certain functions occuring in applied Mathematics were also obtained. The aim of this paper is to obtain the originals of certain image functions with the hope that these results may be useful in electromagnetic diffraction problems, as indicated by F. M. RAGAB ([3], p. 119). Moreover we shall also establish certain relations

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between the generalized LAPLACE transforms of different parameters. These results have been obtained by utilizing the integral representations of the WHITTAKER functions and are stated in the form of two theorems. The theorems may be used to evaluate integrals involving MEIJER's G-function.

When $k + m = \frac{1}{2}$, (1.2) reduces to (1.1) by virtue of the identity

$$x^{m-\frac{1}{2}} W_{\frac{1}{2}-m,m}(x) \equiv e^{-\frac{1}{2}x}.$$

Throughout this paper, the symbol

$$\Phi(p) \underset{k,\, m}{\overset{v}{=\!=}} h(t) \qquad \text{or} \qquad h(t) \underset{k,\, m}{\overset{v}{=\!=}} \Phi(p),$$

will be used to denote (1.2), whilst the notation

$$\Phi(p) \doteqdot h(t) \qquad \text{or} \qquad h(t) \doteqdot \Phi(p)$$

represent the LAPLACE transform.

2. – Here we derive the original of the MEIJER's G-function in the generalised LAPLACE transform. The following results are required in the sequel ([1], pp. 209, 216, 221 and 222; [4], p. 401).

$$x^{\sigma} G^{\alpha\,\beta}_{\gamma\,\delta}\left[x \,\middle|\, \begin{matrix} a_i \\ b_j \end{matrix}\right] = G^{\alpha\,\beta}_{\gamma\,\delta}\left[x \,\middle|\, \begin{matrix} a_i + \sigma \\ b_j + \sigma \end{matrix}\right], \tag{2.1}$$

$$G^{\alpha\,\beta}_{\gamma\,\delta}\left[x^{-1} \,\middle|\, \begin{matrix} a_i \\ b_j \end{matrix}\right] = G^{\beta\,\alpha}_{\delta\,\gamma}\left[x \,\middle|\, \begin{matrix} 1 - b_j \\ 1 - a_i \end{matrix}\right], \tag{2.2}$$

$$G^{2\,0}_{0\,2}(x \mid a,\, b) = 2x^{\frac{1}{2}(a+b)} K_{a-b}(2\sqrt{x}), \tag{2.3}$$

$$G^{2\,0}_{1\,2}\left[x \,\middle|\, \begin{matrix} l - k + 1 \\ l + m + \frac{1}{2},\; l - m + \frac{1}{2} \end{matrix}\right] = x^{l} e^{-\frac{1}{2}x} W_{k,m}(x), \tag{2.4}$$

$$G^{2\,1}_{1\,2}\left[x \,\middle|\, \begin{matrix} k + l + 1 \\ l - m + \frac{1}{2},\; l + m + \frac{1}{2} \end{matrix}\right] = \Gamma(\tfrac{1}{2} - k + m)\, \Gamma(\tfrac{1}{2} - k - m) \times \times x^{l} e^{\frac{1}{2}x} W_{k,m}(x), \tag{2.5}$$

$$G^{1\,2}_{2\,2}\left[x \,\middle|\, \begin{matrix} 1 - a,\; 1 - b \\ 0,\; 1 - c \end{matrix}\right] = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}\, {}_2F_1(a,\, b;\; c;\; -x) \tag{2.6}$$

and

$$(2.7)\qquad G^{\alpha\,\beta}_{\gamma\,\delta}\left[x\,\middle|\,\begin{matrix}a_1, & \dots, & a_\gamma\\ b_1, & \dots, & b_\delta\end{matrix}\right] = (2\pi)^{(1-s)(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)}\times$$

$$\times\, s^{\{\Sigma b_\delta-\Sigma a_\gamma+\frac{1}{2}\gamma-\frac{1}{2}\delta+1\}}\times G^{s\alpha,\, s\beta}_{s\gamma,\, s\delta}\left[x^s\, s^{s(\gamma-\delta)}\,\middle|\,\begin{matrix}\frac{a_1}{s}, & \dots, & \frac{a_\gamma+s-1}{s}\\ \frac{b_1}{s}, & \dots, & \frac{b_\delta+s-1}{s}\end{matrix}\right] (^1),$$

where s is a positive integer.

In a recent paper ([4], p. 402) the author has shown that

$$(2.8)\qquad p^\varrho\, G^{\alpha\,\beta}_{\gamma\,\delta}\left[p^{-n}\,\middle|\,\begin{matrix}a_1, & \dots, & a_\gamma\\ b_1, & \dots, & b_\delta\end{matrix}\right] \underset{k,\ m}{\overset{v}{=\!=}} (2\pi)^{\frac{1}{2}(n-1)}\, n^{\varrho-k-m}\, t^{-\varrho}\times$$

$$\times\, G^{\alpha,\ \beta+n}_{\gamma+n,\ \delta+2n}\left[\frac{t^n}{n^n}\,\middle|\,\begin{matrix}a_1, & \dots, & a_\beta, & \Phi_1, & \dots, & \Phi_n, & a_{\beta+1}, & \dots, & a_\gamma\\ b_1, & \dots, & b_\alpha, & \Psi_1, & \dots, & \Psi_{2n}, & b_{\alpha+1}, & \dots, & b_\delta\end{matrix}\right],$$

where $\qquad \Phi_{i+1} = \dfrac{2\varrho+2k-2m+2i-1}{2n}, \qquad \Psi_{i+1} = \dfrac{\varrho+i}{n};$

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(1) For the sake of brevity, the following abbreviations are used throughout this paper;

$$\sum a_\gamma = a_1 + a_2 + \dots + a_\gamma; \qquad \sum b_\delta = b_1 + b_2 + \dots + b_\delta;$$

$$\Gamma(\alpha\pm\beta) = \Gamma(\alpha+\beta)\,\Gamma(\alpha-\beta);$$

$$G^{h\,l}_{q\,r}\left[x\,\middle|\,\begin{matrix}\dots, & \frac{a_1}{s}, & \dots, & \frac{a_\gamma+s-1}{s}, & \dots\\ \dots, & \frac{b_1}{s}, & \dots, & \frac{b_\delta+s-1}{s}, & \dots\end{matrix}\right] =$$

$$= G^{h\,l}_{q\,r}\left[x\,\middle|\,\begin{matrix}\dots, & \frac{a_1}{s}, & \frac{a_1+1}{s}, & \dots, & \frac{a_1+s-1}{s}, & \dots, & \frac{a_\gamma}{s}, & \frac{a_\gamma+1}{s}, & \dots, & \frac{a_\gamma+s-1}{s}, & \dots\\ \dots, & \frac{b_1}{s}, & \frac{b_1+1}{s}, & \dots, & \frac{b_1+s-1}{s}, & \dots, & \frac{b_\delta}{s}, & \frac{b_\delta+1}{s}, & \dots, & \frac{b_\delta+s-1}{s}, & \dots\end{matrix}\right];$$

$$G^{h\,l}_{q\,r}\left[x\,\middle|\,\begin{matrix}\dots, & a_1\pm b_1, & a_2\pm b_2, & \dots, & a_\gamma\pm b_\gamma, & \dots\\ \dots, & c_1\pm d_1, & c_2\pm d_2, & \dots, & c_\delta\pm d_\delta, & \dots\end{matrix}\right] =$$

$$= G^{h\,l}_{q\,r}\left[x\,\middle|\,\begin{matrix}\dots, & a_1+b_1, & a_1-b_1, & a_2+b_2, & a_2-b_2, & \dots, & a_\gamma+b_\gamma, & a_\gamma-b_\gamma, & \dots\\ \dots, & c_1+d_1, & c_1-d_1, & c_2+d_2, & c_2-d_2, & \dots, & c_\delta+d_\delta, & c_\delta-d_\delta, & \dots\end{matrix}\right].$$

$$\Psi_{n+i+1} = \frac{\varrho - 2m + i}{n} \quad \text{for } i = 0, 1, \ldots, n-1;$$

$$R(\min (nb_h + m \pm m + 1 - \varrho)) > 0 \qquad \text{for } h = 1, 2, \ldots, \alpha; \qquad R(p) > 0;$$

$$|\arg z| < \left[\alpha + \beta - \frac{1}{2}\gamma - \frac{1}{2}\delta - \frac{1}{2}n\right]\pi \quad \text{and} \quad \alpha + \beta > \frac{1}{2}\gamma + \frac{1}{2}\delta + \frac{1}{2}n.$$

On using the property of the MEIJER's G-function (2.7) it can be further written as

$$(2.9) \qquad p^{\varrho}\, G^{\alpha\,\beta}_{\gamma\,\delta}\left[ap^{-n/s}\left|\begin{matrix} a_1, \ldots, a_\gamma \\ b_1, \ldots, b_\delta\end{matrix}\right.\right] \overset{v}{\underset{k,\, m}{=\!=}} t^{-\varrho}\,(2\pi)^{\frac{1}{2}n-\frac{1}{2}}\, n^{\varrho-k-m} \times$$

$$\times\ (2\pi)^{(1-s)(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)}\, s^{\Sigma b_\delta - \Sigma a_\gamma + \frac{1}{2}\gamma - \frac{1}{2}\delta + 1} \times$$

$$\times G^{s\alpha,\, s\beta+n}_{s\gamma+n,\, s\delta+2n}\left[\frac{a^s\, t^n}{s^{s(\delta-\gamma)}\, n^n}\left|\begin{matrix} \frac{a_1}{s}, \ldots, \frac{a_\beta + s - 1}{s}, \Phi_1, \ldots, \Phi_n, \frac{a_\beta + 1}{s}, \ldots, \frac{a_\gamma + s - 1}{s} \\ \frac{b_1}{s}, \ldots, \frac{b_\alpha + s - 1}{s}, \Psi_1, \ldots, \Psi_{2n}, \frac{b_\alpha + 1}{s}, \ldots, \frac{b_\delta + s - 1}{s}\end{matrix}\right.\right] (^2),$$

where $R(\min (nb_h + ms \pm ms + s - s\varrho)) > 0$, $h = 1, 2, \ldots, \alpha$; $R(p) > 0$;

$$|\arg a^s| < \left[s\alpha + s\beta - \frac{1}{2}s\gamma - \frac{1}{2}s\delta - \frac{1}{2}n\right]\pi, \qquad \alpha + \beta > \frac{1}{2}\gamma + \frac{1}{2}\delta + \frac{1}{2}(n/s)$$

and Φ's and Ψ's have the same values as defined in (2.8).

When $n = 1$, (2.9) yields

$$(2.10) \qquad p^{\varrho}\, G^{\alpha\,\beta}_{\gamma\,\delta}\left[ap^{-1/s}\left|\begin{matrix} a_1, \ldots, a_\gamma \\ b_1, \ldots, b_\delta\end{matrix}\right.\right] \overset{v}{\underset{k,\, m}{=\!=}} t^{-\varrho}\,(2\pi)^{(1-s)(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)} \times$$

$$\times\ s^{\Sigma b_\delta - \Sigma a_\gamma + \frac{1}{2}\gamma - \frac{1}{2}\delta + 1} \times G^{s\alpha,\, s\beta+1}_{s\gamma+1,\, s\delta+2}\left[\frac{a^s\, t}{s^{s(\delta-\gamma)}}\left|\begin{matrix} \frac{a_1}{s}, \ldots, \frac{a_\gamma + s - 1}{s}, \varrho + k - m - \frac{1}{2} \\ \frac{b_1}{s}, \ldots, \frac{b_\delta + s - 1}{s}, \varrho . \varrho - 2m\end{matrix}\right.\right],$$

where $R(nb_h + ms \pm ms + s - s\varrho) > 0$ for $h = 1, 2, \ldots, \alpha$; $|\arg a^s| < (s\alpha + s\beta - \frac{1}{2}s\gamma - \frac{1}{2}s\delta - \frac{1}{2})\pi$, $\alpha + \beta > \frac{1}{2}(\gamma + \delta + (n/s))$ and $R(p) > 0$.

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(2) For the behaviour of the MEIJER's G-function, see [1], p. 212.

3. – Some interesting particular cases of (2.9) are given below. The results obtained supply us some new integral representations for the WHITTAKER's function. In what follows n and s are positive integers.

(i) In case we put $\alpha = \delta = 2$, $\beta = \gamma = 0$, $b_1 = \nu/2$, $b_2 = -\nu/2$ in (2.9) and use (2.3), we get

$$(3.1) \qquad 2p^{\varrho} K_{\nu}(ap^{-n/2s}) \underset{k,\ m}{\overset{v}{=\!=}} n^{\varrho-k-m} (2\pi)^{\frac{1}{2}(1+n-2s)} t^{-\varrho} \times$$

$$\times\ G^{2s,\ n}_{n,\ 2s+2n}\left[\frac{t^n a^{2s}}{n^n (2s)^{2s}} \left| \begin{matrix} \Phi_1, \ldots, \Phi_n \\ \pm \dfrac{\nu}{2s}, \ldots, \dfrac{2s \pm \nu - 2}{2s}, \Psi_1, \ldots, \Psi_{2n} \end{matrix} \right.\right],$$

where $R(1 + m \pm m \pm n(\nu/2s) - \varrho) > 0$, $|\arg a^{2s}| < ((2s - n)/2)\pi$, $2s > n$ and $R(p) > 0$.

(ii) Taking $\alpha = \delta = 2$, $\beta = 0$, $\gamma = 1$, $a_1 = 1 - \lambda$, $b_1 = \frac{1}{2} + \mu$, $b_2 = \frac{1}{2} - \mu$ in (2.9) and using the formula (2.4) we find that

$$(3.2) \qquad p^{\varrho} \exp(-\tfrac{1}{2}\, ap^{-n/s})\, W_{\lambda,\mu}(ap^{-n/s}) \underset{k,\ m}{\overset{v}{=\!=}} (2\pi)^{\frac{1}{2}(n-s)} s^{\lambda+\frac{1}{2}} n^{\varrho-k-m} \times$$

$$\times\ t^{-\varrho}\, G^{2s,\ n}_{s+n,\ 2s+2n}\left[\frac{t^n a^s}{n^n s^s} \left| \begin{matrix} \dfrac{1-\lambda}{s}, \ldots, \dfrac{s-\lambda}{s}, \Phi_1, \ldots, \Phi_n \\ \dfrac{1 \pm 2\mu}{2s}, \ldots, \dfrac{2s \pm 2\mu - 1}{2s}, \Psi_1, \ldots, \Psi_{2n} \end{matrix} \right.\right],$$

where $R(n \pm 2n\mu - 2s\varrho + 2ms \pm 2ms + 2s) > 0$, $R(p) > 0$, $|\arg a^s| < ((s - n)/2)\pi$, $s > n$.

On taking $\lambda = 0$, $\mu = \frac{1}{2}$ in (3.2), it gives

$$(3.3) \qquad p^{\varrho} \exp(-ap^{-n/s}) \underset{k,\ m}{\overset{v}{=\!=}} (2\pi)^{\frac{1}{2}(n-s)} s^{\frac{1}{2}} n^{\varrho-k-m} t^{-\varrho} \times$$

$$\times\ G^{s,n}_{n,s+2n}\left[\frac{t^n a^s}{n^n s^s} \left| \begin{matrix} \Phi_1, \ldots, \Phi_n \\ 0, \ldots, (s-1)/s, \Psi_1, \ldots, \Psi_{2n} \end{matrix} \right.\right],$$

where $R(1 - \varrho + m \pm m) > 0$, $R(p) > 0$, $|\arg a^s| < ((s - n)/2)\pi$, $s > n$.

(iii) If we take $\alpha = 2$, $\beta = \gamma = 1$, $\delta = 2$, $a_1 = 1 + \lambda$, $b_1 = \frac{1}{2} - \mu$ and $b_2 = \frac{1}{2} + \mu$ in (2.9) and use (2.5), we obtain

$$(3.4) \qquad p^{\varrho} \exp\left(\tfrac{1}{2} ap^{-n/s}\right) W_{\lambda,\mu}(ap^{-n/s}) \underset{k,\, m}{\overset{v}{=\!=}} \frac{(2\pi)^{\frac{1}{2}(2+n-3s)}\, s^{\frac{1}{2}-\lambda}\, n^{\varrho-k-m}}{t^{\varrho}\, \Gamma(\frac{1}{2} - \lambda \pm \mu)} \times$$

$$\times\; G^{2s,\, s+n}_{s+n,\, 2s+2n}\left[\frac{t^n a^s}{n^n s^s}\middle|\begin{matrix}\frac{1+\lambda}{s}, \ldots, \frac{s+\lambda}{s}, \Phi_1, \ldots, \Phi_n \\ \frac{1 \pm 2\mu}{2s}, \ldots, \frac{2s \pm 2\mu - 1}{2s}, \Psi_1, \ldots, \Psi_{2n}\end{matrix}\right],$$

where $R(n + 2s + 2ms \pm 2ms \pm 2n\mu - 2s\varrho) > 0$, $R(p) > 0$, $|\arg a^s| < ((3s - n)/2)\pi$ and $3s > n$.

(iv) Applying (2.2) and taking $\beta = \gamma = 2$, $\alpha = 0$, $\delta = 1$, $a_1 = \frac{1}{2} - \mu$, $a_2 = \frac{1}{2} + \mu$ and $b_1 = \lambda$, replacing a by $1/a$, we get

$$(3.5) \qquad p^{\varrho} \exp\left(-\tfrac{1}{2} ap^{n/s}\right) W_{\lambda,\mu}(ap^{n/s}) \underset{k,\, m}{\overset{v}{=\!=}} (2\pi)^{\frac{1}{2}(n-s)}\, s^{\lambda+\frac{1}{2}}\, n^{\varrho-k-m} \times$$

$$\times\, t^{-\varrho}\, G^{2s+n,\, 0}_{s+2n,\, 2s+n}\left[\frac{a^s n^n}{s^s t^n}\middle|\begin{matrix}\frac{1-\lambda}{s}, \ldots, \frac{s-\lambda}{s}, \frac{1-\varrho}{n}, \ldots, \frac{n-\varrho}{n}, \frac{1+2m-\varrho}{n}, \ldots, \frac{n+2m-\varrho}{n} \\ \frac{1 \pm 2\mu}{2s}, \ldots, \frac{2s \pm 2\mu - 1}{2s}, \frac{3+2m-2k-2\varrho}{2n}, \ldots, \frac{2n+1+2m-2k-2\varrho}{2n}\end{matrix}\right],$$

where $R(p) > 0$, $|\arg a^s| < ((s - n)/2)\pi$, $s > n$.

For $\lambda = 0$ and $\mu = \frac{1}{2}$ (3.5) yields

$$(3.6) \qquad p^{\varrho} \exp\left(-ap^{n/s}\right) \underset{k,\, m}{\overset{v}{=\!=}} (2\pi)^{\frac{1}{2}(n-s)}\, s^{\frac{1}{2}}\, n^{\varrho-k-m}\, t^{-\varrho} \times$$

$$\times\; G^{s+n,\, 0}_{2n,\, s+n}\left[\frac{a^s n^n}{s^s t^n}\middle|\begin{matrix}\frac{1-\varrho}{n}, \ldots, \frac{n-\varrho}{n}, \frac{1+2m-\varrho}{n}, \ldots, \frac{n+2m-\varrho}{n} \\ 0, \ldots, \frac{s-1}{s}, \frac{3+2m-2k-2\varrho}{2n}, \ldots, \frac{2n+1+2m-2k-2\varrho}{2n}\end{matrix}\right],$$

where $|\arg a^s| < ((s - n)/2)\pi$, $R(p) > 0$, $s > n$.

(v) We now put $\beta = \gamma = 2$, $\alpha = \delta = 1$, $b_1 = -\lambda$, $a_1 = \frac{1}{2} + \mu$, $a_2 = \frac{1}{2} - \mu$ and apply the property of the MEIJER's G-function (2.2), then on using the formula (2.5), it gives

$$(3.7)\qquad p^{\varrho}\exp(\tfrac{1}{2}ap^{n/s})\,W_{\lambda,\mu}(ap^{n/s})\ \underset{k,\,m}{\overset{v}{=\!=}}\ \frac{(2\pi)^{\frac{1}{2}(n+2-3s)}\,s^{\frac{1}{2}-\lambda}\,n^{\varrho-k-m}}{\Gamma(\frac{1}{2}-\lambda\pm\mu)\,t^{\varrho}}\times$$

$$\times G^{2s+n,s}_{s+2n,2s+n}\left[\frac{a^s n^n}{s^s t^n}\left|\begin{matrix}\frac{1+\lambda}{s},\ldots,\frac{s+\lambda}{s},\frac{1-\varrho}{s},\ldots,\frac{n-\varrho}{n},\frac{1+2m-\varrho}{n},\ldots,\frac{n+2m-\varrho}{n}\\ \frac{1\pm2\mu}{2s},\ldots,\frac{2s\pm2\mu-1}{2s},\frac{3+2m-2k-2\varrho}{2n},\ldots,\frac{2n+1+2m-2k-2\varrho}{2n}\end{matrix}\right.\right],$$

where $R(s+ms\pm ms-n\lambda-s\varrho)>0$, $|\arg a^s|<((3s-n)/2)\pi$, $3s>n$ and $R(p)>0$.

(vi) Lastly take $a_1=1$, $a_2=\nu$, $b_1=\lambda$, $b_2=\mu$, $\beta=1$, $\alpha=\gamma=\delta=2$ in (2.9), replace a by $1/a$ and use the relations (2.2) and (2.6), we then obtain

$$(3.8)\qquad p^{\varrho}\,{}_2F_1(\lambda,\mu;\nu;-ap^{n/s})\ \underset{k,\,m}{\overset{v}{=\!=}}\ \frac{\Gamma(\nu)\,(2\pi)^{\frac{1}{2}(n+1-2s)}\,s^{\lambda+\mu-\nu}\,n^{\varrho-k-m}}{\Gamma(\lambda)\,\Gamma(\mu)\,t^{\varrho}}\times$$

$$\times G^{s+n,2s}_{2s+2n,2s+n}\left[\frac{a^s n^n}{t^n}\left|\begin{matrix}\frac{1-\lambda}{s},\ldots,\frac{s-\lambda}{s},\frac{1-\mu}{s},\ldots,\frac{s-\mu}{s},\frac{1-\varrho}{n},\ldots,\frac{n-\varrho}{n},\frac{1+2m-\varrho}{n},\ldots,\frac{n+2m-\varrho}{n}\\ 0,\ldots,\frac{s-1}{s},\frac{3+2m-2k-2n\varrho}{2n},\ldots,\frac{2n+1+2m-2k-2n\varrho}{2n},\frac{1-\nu}{s},\ldots,\frac{s-\nu}{s}\end{matrix}\right.\right],$$

where $R(s+n\lambda+ms\pm ms-s\varrho)>0$, $R(p)>0$, $|\arg a^s|<((2s-n)/2)\pi$, $2s>n$, $R(s+n\mu+ms\pm ms-s\varrho)>0$.

When $\mu=\nu$, (3.8) reduces to

$$(3.9)\qquad p^{\varrho}\,(1+ap^{n/s})^{-\lambda}\ \underset{k,\,m}{\overset{v}{=\!=}}\ \frac{(2\pi)^{\frac{1}{2}(n+1-2s)}}{\Gamma(\lambda)}\,s^{\lambda}\,n^{\varrho-k-m}\,t^{-\varrho}\times$$

$$\times G^{s+n,s}_{s+2n,s+n}\left[\frac{a^s n^n}{t^n}\left|\begin{matrix}\frac{1-\lambda}{s},\ldots,\frac{s-\lambda}{s},\frac{1-\varrho}{n},\ldots,\frac{n-\varrho}{n},\frac{1+2m-\varrho}{n},\ldots,\frac{n+2m-\varrho}{n}\\ 0,\ldots,\frac{s-1}{s},\frac{3+2m-2k-2\varrho}{2n},\ldots,\frac{2n+1+2m-2k-2\varrho}{2n}\end{matrix}\right.\right],$$

where $R(s-s\varrho+ms\pm ms+n\lambda)>0$, $R(p)>0$, $|\arg a^s|<((2s-n)/2)\pi$, $2s>n$.

4. - Theorem I.

If

$$\Phi(p) \underset{k,\ m}{\overset{v}{=\!=}} h(t), \tag{4.1}$$

and

$$\Psi(p) \underset{\lambda,\ \mu}{\overset{v}{=\!=}} t^{(s\varrho/n)-1}\, h(t^{s/n}),$$

then

$$\Psi(p) = (2\pi)^{\frac{1}{2}(n-s)}\, s^{\lambda+\mu+(s/n)+(s\varrho/n)-1}\, n^{-k-m}\, p^{1-(s\varrho/n)-(s/n)} \times \tag{4.2}$$

$$\times \int_0^\infty G^{2s+n,0}_{s+2n,2s+n}\left[\frac{p^s\, n^n}{s^s\, t^n}\,\middle|\, \begin{matrix}\alpha_1, & \ldots, & \alpha_{s+2n}\\ \beta_1, & \ldots, & \beta_{2s+n}\end{matrix}\right] \Phi(t)\, \mathrm{d}t,$$

provided that the integral is convergent, $R(p) > 0$, $s > n$ and the generalized LAPLACE transforms of $|h(t)|$ and $|t^{(s\varrho/n)-1}\, h(t^{s/n})|$ exist, where

$$\alpha_{\nu+1} = \frac{n + 2n\mu + 2s + 2s\varrho - 2n\lambda + 2n\nu}{2ns},$$

$$\alpha_{s+i+1} = \frac{2+i}{n}, \qquad \alpha_{s+n+i+1} = \frac{2m+2+i}{n},$$

$$\beta_{\nu+1} = \frac{s + 2n\mu + s\varrho + n\nu}{ns}, \qquad \beta_{s+\nu+1} = \frac{s + s\varrho + n\nu}{ns},$$

$$\beta_{2s+i+1} = \frac{5 + 2m - 2k + 2i}{2n},$$

for $i = 0, 1, \ldots, n-1$ and $\nu = 0, 1, \ldots, s-1$.

Proof: Applying the PARSEVAL-GOLDSTEIN theorem of the generalized LAPLACE transform, namely if

$$\Phi_1(p) \underset{k,\ m}{\overset{v}{=\!=}} h_1(x) \quad \text{and} \quad \Phi_2(p) \underset{k,\ m}{\overset{v}{=\!=}} h_2(x),$$

then

$$\int_0^\infty \Phi_1(x)\, h_2(x)\, x^{-1}\, \mathrm{d}x = \int_0^\infty h_1(x)\, \Phi_2(x)\, x^{-1}\, \mathrm{d}x, \tag{4.3}$$

to the relations (4.1) and (3.5), replacing ϱ by $\dfrac{2s\varrho + 2n\mu - n}{2s}$, we obtain after a little simplifications

$$\int_0^\infty t^{(2n\mu+2s\varrho-n-2s)/2s} \exp(-\tfrac{1}{2}\, at^{n/s})\; W_{\lambda,\mu}(at^{n/s})\, h(t)\, dt =$$

$$= (2\pi)^{\frac{1}{2}(n-s)}\, s^{\lambda+\mu+(s\varrho/n)+(s/n)}\, a^{\frac{1}{2}-(s/n)-(\mu/n)-(s\varrho/n)}\, n^{-k-m-1} \times$$

$$\times \int_0^\infty G^{2s+n,0}_{s+2n,2s+n}\left[\frac{a^s\, n^n}{s^s\, t^n}\,\middle|\, \begin{matrix}\alpha_1, \ldots, \alpha_{s+2n}\\ \beta_1, \ldots, \beta_{2s+n}\end{matrix}\right] \Phi(t)\, dt\,.$$

We now put $t = u^{s/n}$ in the integral on the left, so that

$$\int_0^\infty u^{(s\varrho/n)+\mu-(3/2)}\, e^{-\frac{1}{2}au}\, W_{\lambda,\mu}(au)\, h(u^{s/n})\, du =$$

$$= (2\pi)^{\frac{1}{2}(n-s)}\, s^{\lambda+\mu+(s/n)+(s\varrho/n)-1}\, a^{\frac{1}{2}-(s/n)-\mu-(s\varrho/n)} \times$$

$$\times\, n^{-k-m} \int_0^\infty G^{2s+n,0}_{s+2n,2s+n}\left[\frac{a^s\, n^n}{s^s\, t^n}\,\middle|\, \begin{matrix}\alpha_1, \ldots, \alpha_{s+2n}\\ \beta_1, \ldots, \beta_{2s+n}\end{matrix}\right] \Phi(t)\, dt\,.$$

The theorem follows immediately from above, on multiplying both sides by $a^{\mu+\frac{1}{2}}$ and replacing a by p.

Some interesting particular cases of the theorem are given below:

Corollary I.

If

$$\Phi(p) \doteqdot h(t)$$

and

$$\Psi(p) \underset{\lambda,\ \mu}{\overset{v}{=\!=}} t^{(s\varrho/n)-1}\, h(t^{s/n}),$$

then

$$\Psi(p) = (2\pi)^{\frac{1}{2}(n-s)}\, s^{\lambda+\mu+(s/n)+(s\varrho/n)-1}\, n^{-\frac{1}{2}}\, p^{1-(s\varrho/n)-(s/n)} \times \tag{4.4}$$

$$\times \int_0^\infty \Phi(t)\; G^{2s,0}_{s+n,2s}\left[\frac{p^s\, n^n}{s^s\, t^n}\,\middle|\, \begin{matrix}\alpha_1, \ldots, \alpha_{s+n}\\ \beta_1, \ldots, \beta_{2s}\end{matrix}\right] dt,$$

under the conditions of the theorem.

Corollary II.

If

$$\Phi(p) \underset{k,\ m}{\overset{v}{=\!=}} h(t)$$

and

$$\Psi(p) \doteqdot t^{(s\varrho/n)-1}\, h(t^{s/n}),$$

then

$$\Psi(p) = (2\pi)^{\frac{1}{2}(n-s)}\, s^{(s/n)+(s\varrho/n)-\frac{1}{2}}\, n^{-k-m}\, p^{1-(s\varrho/n)-(s/n)} \times \tag{4.5}$$

$$\times \int_0^\infty \Phi(t)\; G^{s+n,0}_{2n,s+n}\left[\frac{p^s\ n^n}{s^s\ t^n}\left|\begin{matrix}\frac{2}{n}, \ldots, \frac{n+1}{n}, \frac{2m+2}{n}, \ldots, \frac{2m+n+1}{n} \\ \frac{1+\varrho}{n}, \frac{1+\varrho}{n}+\frac{1}{s}, \ldots, \frac{1+\varrho}{n}+\frac{s-1}{s}, \frac{5+2m-2k}{2n}, \ldots, \frac{2n+3+2m-2k}{2n}\end{matrix}\right.\right] \mathrm{d}t,$$

under the conditions of the theorem.

Corollary I and Corollary II can be derived from the theorem on taking $k+m=\frac{1}{2}$ and $\lambda+\mu=\frac{1}{2}$ respectively.

A particular case of the theorem, when $k+m=\frac{1}{2}$ and $\lambda+\mu=\frac{1}{2}$ was recently obtained by the author in an earlier paper ([**6**], p. 60).

5. – Theorem II.

If

$$\Phi(p) \underset{k,\ m}{\overset{v}{=\!=}} h(t)$$

and

$$\Psi(p) \underset{\lambda,\ \mu}{\overset{v}{=\!=}} t^{(s\varrho/n)-1}\, h(t^{-s/n}),$$

then

$$\Psi(p) = (2\pi)^{\frac{1}{2}(n-s)}\, s^{\lambda+\mu+(s\varrho/n)-(s/n)-1}\, p^{1+(s/n)-(s\varrho/n)} \times \tag{5.1}$$

$$\times\, n^{-k-m}\int_0^\infty G^{2s,n}_{s+n,2s+2n}\left[\frac{t^n\ p^s}{n^n\ s^s}\left|\begin{matrix}\gamma_1, \ldots, \gamma_{s+n} \\ \delta_1, \ldots, \delta_{2s+2n}\end{matrix}\right.\right] \Phi(t)\,\mathrm{d}t,$$

where $$\gamma_{\nu+1} = \frac{n+2n\mu-2n\lambda+2s\varrho-2s+2n\nu}{2ns},$$

$$\gamma_{s+i+1} = \frac{2k - 2m - 3 + 2i}{2n}, \qquad \delta_{\nu+1} = \frac{s\varrho + 2n\mu - s + n\nu}{ns},$$

$$\delta_{s+\nu+1} = \frac{s\varrho - s + n\nu}{ns}, \qquad \delta_{2s+i+1} = \frac{i-1}{n}, \qquad \delta_{2s+n+i+1} = \frac{i - 2m - 1}{n},$$

for $i = 0, 1, \ldots, n-1$ and $\nu = 0, 1, \ldots, s-1$; provided that the integral is convergent, $R(p) > 0$ and the generalized LAPLACE transforms of $|h(t)|$ and $|t^{(s\varrho/n)-1} h(t^{-s/n})|$ exist, $s > n$ and $R(n\mu \pm n\mu + s\varrho + ms \pm ms + s) > 0$.

The proof of the Theorem II follows on the same lines as that of Theorem I, if we use (3.2) instead of (3.5).

We now give some interesting particular cases of the theorem.

Corollary I.

If

$$\Phi(p) \doteqdot h(t)$$

and

$$\Psi(p) \underset{\lambda,\ \mu}{\overset{v}{=\!=}} t^{(s\varrho/n)-1} h(t^{-s/n}),$$

then

$$\Psi(p) = (2\pi)^{\frac{1}{2}(2-n-s)} n^{-\frac{1}{2}} s^{\lambda+\mu+(s\varrho/n)-(s/n)-1} p^{1+(s/n)-(s\varrho/n)} \times \tag{5.2}$$

$$\times \int_0^\infty \Phi(t)\ G^{2s,0}_{s,2s+n}\left[\frac{t^n p^s}{n^n s^s}\middle|\begin{matrix}\gamma_1, \ldots, \gamma_s\\ \delta_1, \ldots, \delta_{2s+n}\end{matrix}\right] dt,$$

under the conditions of the theorem.

Put $k + m = \frac{1}{2}$ in (5.1), then we obtain Cor. I.

Corollary II.

If

$$\Phi(p) \underset{k,\ m}{\overset{v}{=\!=}} h(t)$$

and

$$\Psi(p) \doteqdot t^{(s\varrho/n)-1} h(t^{-s/n}),$$

then

$$\Psi(p) = (2\pi)^{\frac{1}{2}(2-n-s)} s^{(s\varrho/n)-(s/n)-\frac{1}{2}} p^{(s/n)-(s\varrho/n)+1} \times \tag{5.3}$$

$$\times n^{-(k+m)} \int_0^\infty \Phi(t)\ G^{s,n}_{n,s+2n}\left[\frac{t^n p^s}{n^n s^s}\middle|\begin{matrix}\frac{2k-2m-3}{2n}, \ldots, \frac{2n+2k-2m-5}{2n}\\ \frac{\varrho-1}{n}, \ldots, \frac{\varrho-1}{n}+\frac{s-1}{s}, -\frac{1}{n}, \ldots, \frac{n-2}{n}, -\frac{2m+1}{n}, \ldots, \frac{n-2m-2}{n}\end{matrix}\right] dt,$$

under the conditions of the theorem.

To obtain Cor. II put $\lambda + \mu = \frac{1}{2}$ in the theorem.

Corollary III. When $k + m = \frac{1}{2}$ and $\lambda + \mu = \frac{1}{2}$, then the theorem can be enunciated in the following form:

If

$$\Phi(p) \doteqdot h(t)$$

and

$$\Psi(p) \doteqdot t^{(s\varrho/n)-1}\, h(t^{-s/n}),$$

then

$$\Psi(p) = (2\pi)^{\frac{1}{2}(2-n-s)}\, s^{(s\varrho/n)-(s/n)-\frac{1}{2}}\, p^{1+(s/n)-(s\varrho/n)}\, n^{-\frac{1}{2}} \times \tag{5.4}$$

$$\times \int_0^\infty \Phi(t)\, G^{s,0}_{0,s+n}\left[\frac{t^n\, p^s}{n^n\, s^s}\,\middle|\, \frac{\varrho-1}{n}, \ldots, \frac{\varrho-1}{n} + \frac{s-1}{s}, -\frac{1}{n}, \ldots, \frac{n-2}{n}\right] dt,$$

under the conditions of the theorem.

When $n \to 1$ and $s \to 1$ in Cor. III, it reduces to a result given by H. C. Gupta ([2], p. 142).

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