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Steady Laminar Flow of a Reiner-Rivlin Fluid in an Annulus, in the Presence of a Radial Magnetic Field. (**)

1. - Introduction.

Let us consider non-conducting and infinitely long coaxial circular cylinders with inner radius a and outer radius b. Let us take the z axis along the axis of the annulus in the direction of flow and take the origin at a fixed point on this axis and construct a right-handed system of cylindrical polar coordinates. Coordinates of a typical point in the channel are (r, θ, z). A radial magnetic field, with intensity ω/r (ω is a constant) at any point (r, θ, z) emanating radially from the axis is assumed to exist. We consider the steady flow of a REINER-RIVLIN conducting incompressible fluid of constant electric and magnetic properties in the channel described above, when there exists a constant pressure gradient in the direction of the axis. The rheological behaviour exhibited by non-Newtonian visco-inelastic fluids can be adequately studied by generalising the stress-strain velocity relations, of classical hydrodynamics in a manner suggested by RIVLIN [2] and REINER [3], i.e. by introducing second order terms of the type $d^i_\alpha\, d^\alpha_j$ (where d^i_j is the rate of deformation tensor). Thus for a REINER-RIVLIN visco-inelastic incompressible conducting fluid the non-linear stress-strain relation is:

$$t^i_j = -p\,\delta^i_j + F_1\, d^i_j + F_2\, d^i_\alpha\, d^\alpha_j, \tag{1}$$

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where though F_1 and F_2 are, in general, functions of material invariants, we shall consider the cases where $F_1 = \mu = \varrho\,\nu =$ constant and $F_2 = \mu_c = \varrho\,\nu_c =$ = constant (though in general these may be variables but we do not consider such cases here), where μ and μ_c are the coefficients of viscosity and cross-viscosity respectively and ν and ν_c are the corresponding kinematic coefficients.

Also $d_{ij} = V_{i,j} + V_{j,i}$ is the rate of deformation tensor (the usual tensorial notation has been used).

2. - The governing equations.

Neglecting displacement currents and free charges, regarding LORENTZ force as the only body force in the field and employing c.g.s. electromagnetic units, the equations of hydromagnetic flow are:

$$\varepsilon^{pqi} H_{p,q} = -4\pi J^i\,, \tag{2}$$

$$\varepsilon^{pqi} E_{p,q} = \mu_e \frac{\partial H^i}{\partial t}\,, \tag{3}$$

$$H^i_{,i} = 0\,, \tag{4}$$

$$J^i = \sigma\,(E^i + \mu_e\,\varepsilon^{pqi}\,V_p\,H_q)\,, \tag{5}$$

$$V^i_{,i} = 0\,, \tag{6}$$

$$\varrho\left[\frac{\partial V^i}{\partial t} + V^i_{,j}\,V^j\right] = t^{ij}_{,j} + \mu_e\,\varepsilon^{pqi}\,J_p\,H_q. \tag{7}$$

Eliminating J^i and E^i from (2), (3), and (5) and J^i in (2) and (7), we have respectively:

$$\frac{\partial H^i}{\partial t} = -\,\varepsilon^{rsi}\,(\varepsilon_{pqr}\,V^p\,H^q)_{,s} + \frac{\lambda}{\sqrt{|g|}}\,(\sqrt{|g|}\,g^{jk}\,H^i_{,k})_{,j} \tag{8}$$

and

$$\varrho\left[\frac{\partial V^i}{\partial t} + V^i_{,j}\,V^j\right] = t^{ij}_{,j} - \frac{\mu_e}{8\pi}\,(\sum_j (H^j)^2)_{,p}\,g^{pi} + \frac{\mu_e}{4\pi}\,H^i_{,j}\,H^j, \tag{9}$$

where $V^i_{,j}$ indicates the covariant derivative of the the velocity vector V^i with respect to a coordinate x^j, $H^i_{,j}$ the covariant derivative of the magnetic field intensity vector with respect to the coordinate x^j, ϱ is the density, μ_e the permeability, σ the conductivity (electrical) and $\lambda = 1/(4\pi\mu_e\sigma)$ the magnetic diffusivity of the fluid. $t^{ij}_{,j}$ is the covariant derivative or the rate of stress with respect to the space coordinate x^j (using the summation convention). J^i is the contravariant current density vector, E_i is the covariant electric field intensity vector. In order to suit the geometry of the problem these equations must be transformed to cylindrical polar coordinates. There are no free charges and no externally applied electric field. The flow is laminar and steady, and on account of it being axisymmetric $V_\theta = V_r = 0$, $\partial V_\theta/\partial t = \partial V_z/\partial t = 0$. From the equation of continuity (6), $\partial V_z/\partial z = 0$ and therefore $V_z = V_z(r)$. As in an earlier paper of one of the authors (AGRAWAL [4]), or following GLOBE [1], [5] or KAPUR and JAIN [6] it can be shown that $H_\theta = 0$; $J_r = J_z = 0$ and the induced current having only a θ component behaves like a solenoidal current. Assuming H_z to be either of the forms $R(r) + Z(z)$ or $R(r)\,Z(z)$ and following GLOBE [1], [5] or KAPUR and JAIN [6] it can be shown that $\partial H_z/\partial z = 0$ and, from equation (4), $H_r = \omega/r$ is the applied magnetic field and $H_z = H_z(r)$.

Transforming equations (8) and (9) to cylindrical polar coordinates, and simplifying with the help of the said results in the preceding paragraph, we have:

$$\frac{\partial p}{\partial r} + \frac{\mu_e}{4\pi}\frac{\mathrm{d}H_z^2}{\mathrm{d}r} = \mu_e\,\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left[\frac{\partial V_z}{\partial r}\right]^2\right\}, \tag{10}$$

$$\frac{\mu_e\,\omega}{4\pi\varrho r}\frac{\mathrm{d}H_z}{\mathrm{d}r} = \frac{1}{\varrho}\frac{\partial p}{\partial z} - \nu\left[\frac{\mathrm{d}^2V_z}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}V_z}{\mathrm{d}r}\right], \tag{11}$$

$$-\frac{\omega}{r}\frac{\mathrm{d}V_z}{\mathrm{d}r} = \lambda\left[\frac{\mathrm{d}^2H_z}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}H_z}{\mathrm{d}r}\right]. \tag{12}$$

(Total derivatives have replaced the partial derivatives because V_z and H_z are functions of r only).

Since the pressure gradient in the axial direction have been taken to be constant we can write:

$$\frac{\partial p}{\partial z} = -P = \text{constant}. \tag{13}$$

From (10) and (13) on integration:

$$Pz + p + \frac{\mu_e}{8\pi}H_z^2 - \mu_c\int\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left[\frac{\partial V_z}{\partial r}\right]^2\right\}\mathrm{d}r = \text{constant} \tag{14}$$

if ν_c is constant, so that once V_z and H_z are determined we can find the pressure at any point in the channel (to the extent of an arbitrary constant) and the pressure variation in the channel.

The boundary conditions in this case are identical with those for Newtonian fluids:

No slip condition at the boundaries:

$$V_z(a) = 0, \qquad V_z(b) = 0; \tag{15}$$

Continuity of the tangential component of the magnetic field:

$$H_z(b) = 0\,. \tag{16}$$

Since J_θ is zero in the wall (non-conducting) and at the fluid wall interface:

$$(J_\theta)_{r=b} = -\frac{1}{4\,\pi}\left[\frac{\partial H_z}{\partial r}\right]_{r=b} = \mu_e\,\sigma\,(V_z\,H_r - V_r\,H_z)_{r=b} = 0. \tag{17}$$

It is interesting to note that the coupled equations (11) and (12) and also the boundary conditions (15), (16), and (17) are same as those for Newtonian fluids (GLOBE [1]). Thus the velocity and magnetic field in this case are independent of cross-viscosity, *whereas the pressure is not*. The values for magnetic field H_z and velocity V_z are same as those for the Newtonian fluids is a corresponding problem and therefore the velocity profiles and magnetic field profiles also are independent of cross-viscosity (since the values of V_z and H_z are same as in GLOBE's case [1] we do not reproduce them). It has been observed that if the walls of this channel are porous the velocity and magnetic field also depend on the cross-viscosity.

3. - Limiting forms of the flow.

(1) The HARTMANN flow of a non-Newtonian fluid can be derived as a limiting case of this by making $a \to \infty$ and $b \to \infty$ such that $b - a$ remains finite say $2\,h$. The expressions for velocity and magnetic fileds remain the same as for Newtonian fluids i.e. those in the analysis of HARTMANN flow (T. G. COWLING [7]) (after setting $E = 0$). Thus the velocity and magnetic field in HARTMANN flow of a non-Newtonian fluid are independent of the cross-viscosity, the pressure in this case also is affected by cross-viscosity.

(2) For non-conducting fluid ($\sigma = 0$) there is no interaction in the flow and field and therefore in the limiting case $\sigma \to 0$, $H_z \to 0$ and velocity

and pressure can be obtained for the corresponding hydrodynamic flow. The pressure in this case is given by:

$$P_z + p + \mu_c \int \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \left[\frac{\partial V_z}{\partial r} \right] \right\} \mathrm{d}r = \text{constant}$$

and therefore pressure field is affected by cross-viscosity whereas the velocity is not.

(3) By making $\nu_c \to \infty$ in this problem and the limiting cases (1) and (2) the results for the corresponding flows of Newtonian fluids can be obtained.

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Summary.

Globe [1] has studied the flow of a Newtonian (viscous incompressible) fluid in the space between two coaxial infinitely long circular cylinders in the presence of a radial magnetic field. The corresponding problem for a non-Newtonian visco-inelastic (Reiner-Rivlin) fluid has been studied in this paper. It has been found that for constant values of cross-viscosity and viscosity, the velocity and magnetic field retain the same value as in Globe's case, but the pressure field is affected by cross-viscosity.

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