Riv. Mat. Univ. Parma (2) 6 (1965), 225-237

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On the Order and Type of Entire Functions. (**)

1. - Introduction.

Let $F(z) = \sum_{n=0}^{\infty} A_n z^n$ be an entire function of order $\varrho\ (0 \leqslant \varrho \leqslant \infty)$ and lower order $\lambda\ (0 \leqslant \lambda \leqslant \infty)$, then

$$(1.1)\qquad \overline{\underline{\mathrm{Lt}}}_{n\to\infty} \frac{n \log n}{\log | A_n |^{-1}} = \begin{cases} \varrho \\ \lambda \end{cases};$$

the lower limit being only true if (see SHAH [3], p. 1047) $| A_n/A_{n+1} |$ is a non-decreasing function of n for $n > n_0$. Further, if T, t be respectively the type and lower type of $F(z)$, then

$$(1.2)\qquad \overline{\underline{\mathrm{Lt}}}_{n\to\infty} \left\{ \frac{n}{\varrho e} | A_n |^{\varrho/n} \right\} = \begin{cases} T \\ t \end{cases};$$

the lower limit being only true if $| A_n/A_{n+1} |$ is a non-decreasing function of n for $n > n_0$ (see SHAH [4], p. 45). For the existence of type T see BOAS ([1], p. 11). We shall throughout suppose that T and t are positive and finite real numbers. A number of relations between two or more entire functions, connecting their orders, lower orders, types and lower types have been recently obtained by a few authors. We, however, wish to state one of their theorems: in fact, S. N. SRIVASTAVA ([7], p. 274) has recently proved:

Theorem A. *If $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ be entire functions of the same order $\varrho\ (0 < \varrho < \infty)$, lower types $t_1\ (0 < t_1 < \infty)$, $t_2\ (0 < t_2 < \infty)$*

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respectively and $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ *be non-decreasing functions for* $n > n_0$, *then the function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where* $|c_n| \sim |\sqrt{|a_n||b_n|}|$, *is an entire function of order* ϱ *and lower type* t *such that*

$$t \geqslant \sqrt{t_1 t_2}\,.$$

It is natural to think of as to what will happen of t, that is, what connection t will have with t_1 and t_2 if $f_1(z)$ and $f_2(z)$ are of orders ϱ_1 and ϱ_2 respectively. The answer is contained in Theorem 1 below, where we assume a different condition, though strong in some sense, connecting the asymtotic behaviour of a_n, b_n and c_n, than Theorem A. Besides, we have obtained several other theorems which furnish the connections between ϱ_1, ϱ_2 and ϱ; T_1, T_2 and T, etc. etc..

2. – Theorem 1. *If* $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *be entire functions of orders* ϱ_1 $(0 < \varrho_1 < \infty)$ *and* ϱ_2 $(0 < \varrho_2 < \infty)$; *lower types* t_1 $(0 < t_1 < \infty)$, t_2 $(0 < t_2 < \infty)$ *respectively, and if* $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ *be non-decreasing functions of* n *for* $n > n_0$, *then the function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *where* (i) $\log|c_n|^{-1} \sim |(\log|a_n|^{-1})^{\alpha} (\log|b_n|^{-1})^{\beta}|$, $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta = 1$, (ii) $|c_n/c_{n+1}|$ *is a non-decreasing function of* n *for* $n > n_0$, *is also an entire function of order* ϱ *and lower type* t *such that*

$$\varrho \leqslant \varrho_1^{\alpha} \varrho_2^{\beta}\,, \tag{2.1}$$

$$t \geqslant t_1^{\alpha} t_2^{\beta}\,. \tag{2.2}$$

Proof. First, we prove that $f(z)$ is an entire function. Since $f_1(z)$ and $f_2(z)$ are entire functions, hence

$$\underset{n\to\infty}{\underline{\mathrm{Lt}}}\ |a_n|^{-1/n} = \underset{n\to\infty}{\underline{\mathrm{Lt}}}\ |b_n|^{-1/n} = \infty\,, \tag{2.3}$$

therefore, for every $\varepsilon > 0$,

$$\begin{cases} (\log|a_n|^{-1})^{\alpha} > (n\log(R-\varepsilon))^{\alpha}, & n > n_1 \\ (\log|b_n|^{-1})^{\beta} > (n\log(R-\varepsilon))^{\beta}, & n > n_2\,. \end{cases} \tag{2.4}$$

Hence using condition (i) of the Theorem, we find that for n sufficiently large

$$\log |c_n|^{-1} > n \log(R - \varepsilon),$$

which means that $f(z)$ is an entire function. Again $f_1(z)$ is of order ϱ, hence from (1.1) we have

$$\frac{\log |a_n|^{-1}}{n \log n} > \frac{1}{\varrho_1 + \varepsilon}, \qquad n > n_1, \ \varepsilon > 0 .$$

Or, we have

$$(\log |a_n|^{-1})^{\alpha} > \left\{ \frac{n \log n}{\varrho_1 + \varepsilon} \right\}^{\alpha}, \qquad n > n_1, \ \varepsilon > 0 .$$

Similarly for $f_2(z)$ we have

$$(\log |b_n|^{-1})^{\beta} > \left\{ \frac{n \log n}{\varrho_2 + \varepsilon} \right\}^{\beta}, \qquad n > n_2, \ \varepsilon > 0 .$$

Therefore, for $n > \max(n_1, n_2)$,

$$(\log |a_n|^{-1})^{\alpha} (\log |b_n|^{-1})^{\beta} > \frac{n \log n}{(\varrho_1 + \varepsilon)^{\alpha} (\varrho_2 + \varepsilon)^{\beta}},$$

since $\alpha + \beta = 1$. But

$$\log |c_n|^{-1} \sim (\log |a_n|^{-1})^{\alpha} (\log |b_n|^{-1})^{\beta} .$$

Therefore, for n sufficiently large,

$$\frac{\log |c_n|^{-1}}{n \log n} > \frac{1}{(\varrho_1 + \varepsilon)^{\alpha} (\varrho_2 + \varepsilon)^{\beta}},$$

which means that

$$\frac{1}{\varrho} = \underset{n\to\infty}{\underline{\mathrm{Lt}}} \frac{\log |c_n|^{-1}}{n \log n} \geqslant \frac{1}{\varrho_1^{\alpha} \varrho_2^{\beta}},$$

and this proves (2.1). We turn to the proof of (2.2). Using (1.2) for $f_1(z)$ and $f_2(z)$, we have

$$\frac{n}{e\,\varrho_1} \mid a_n \mid^{\varrho_1/n} > t_1 - \varepsilon_1, \qquad n > n_1, \tag{2.5}$$

$$\frac{n}{e\,\varrho_2} \mid b_n \mid^{\varrho_2/n} > t_2 - \varepsilon_2, \qquad n > n_2. \tag{2.6}$$

From (2.5) and (2.6) we have

$$(\log \mid a_n \mid^{-1})^{\alpha} < \left[\frac{n}{\varrho_1} \log \frac{n}{e\varrho_1 (t_1 - \varepsilon_1)}\right]^{\alpha}, \qquad n > n_1,$$

$$(\log \mid b_n \mid^{-1})^{\beta} < \left[\frac{n}{\varrho_2} \log \frac{n}{e\varrho_2 (t_2 - \varepsilon_2)}\right]^{\beta}, \qquad n > n_2,$$

and so for n sufficiently large, we have from these two preceding inequalities after having multiplied them together

$$(\log \mid a_n \mid^{-1})^{\alpha} (\log \mid b_n \mid^{-1})^{\beta} < \frac{n}{\varrho_1^{\alpha}\,\varrho^{\beta}} \left(\log \frac{n}{A}\right)^{\alpha} \left(\log \frac{n}{B}\right)^{\beta},$$

where

$$A = e\,\varrho_1 (t_1 - \varepsilon_1), \qquad B = e\,\varrho_2 (t - \varepsilon_2).$$

Thus if

$$\log \mid c_n \mid^{-1} \sim (\log \mid a_n \mid^{-1})^{\alpha} (\log \mid b_n \mid^{-1})^{\beta},$$

we have, for sufficiently large n,

$$\log \mid c_n \mid^{-1} < \frac{n}{\varrho_1^{\alpha}\,\varrho_2^{\beta}} \left(\log \frac{n}{A}\right)^{\alpha} \left(\log \frac{n}{B}\right)^{\beta}. \tag{2.7}$$

But from (2.1) $\varrho_1^{-\alpha}\,\varrho_2^{-\beta} \leqslant \varrho^{-1}$, and so for large n,

$$\log|c_n|^{-1} < \frac{n}{\varrho}(\log n - \log A)^{\alpha}(\log n - \log B)^{\beta}$$

$$= \frac{n}{\varrho}\left(1 - \frac{\log A}{\log n}\right)^{\alpha}\left(1 - \frac{\log B}{\log n}\right)^{\beta}\log n$$

$$< \frac{n}{\varrho}\left\{1 - \frac{\alpha\log A}{\log n} + O\,((\log n)^{-2})\right\}\left\{1 - \frac{\beta\log B}{\log n} + O((\log n)^{-2})\right\}\log n$$

$$= \frac{n}{\varrho}\left[1 - \frac{\log(A^{\alpha}B^{\beta})}{\log n} + O((\log n)^{-2})\right]\log n\,.$$

Therefore for large n

$$1/|c_n|^{\varrho/n} < n^{\left\{1 - \frac{\log(A^{\alpha}B^{\beta})}{\log n} + O((\log n)^{-2})\right\}}$$

or

$$\frac{\varrho\, e\, n\,|c_n|^{\varrho/n}}{\varrho\, e} > n^{\left\{\frac{\log(A^{\alpha}B^{\beta})}{\log n} + O((\log n)^{-2})\right\}},$$

and since

$$\operatorname*{Lt}_{n\to\infty} n^{\left\{\frac{\log(A^{\alpha}B^{\beta})}{\log n} + O((\log n)^{-2})\right\}} = A'^{\alpha}\,B'^{\beta},$$

where

$$A' = e\,\varrho_1\,t_1, \qquad B' = e\,\varrho_2\,t_2,$$

we find that

$$\varrho\, e\, t \geqslant e^{\alpha+\beta}\,t_1^{\alpha}\,t_2^{\beta}\,\varrho_1^{\alpha}\,\varrho_2^{\beta},$$

and using (2.1) again, we have finally

$$t \geqslant t_1^{\alpha}\,t_2^{\beta},$$

which proves (2.2).

We also prove:

Theorem 1'. *If* $f_1(z)=\sum_{n=0}^{\infty} a_n z^n$ *and* $f_2(z)=\sum_{n=0}^{\infty} b_n z^n$ *be two entire functions having the same order* ϱ $(0<\varrho<\infty)$, *lower types* t_1 $(0<t_1<\infty)$, t_2 $(0<t_2<\infty)$, *respectively and if* $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ *be non-decreasing functions of* n *for* $n>n_0$, *then* $f(z)=\sum_{n=0}^{\infty} c_n z^n$, *where* (i) $|c_n|\sim||a_n|^{\alpha}|b_n|^{\beta}|$, $0<\alpha<1$, $0<\beta<1$, $\alpha+\beta=1$, *and* (ii) $|c_n/c_{n+1}|$ *is a non-decreasing function of* n *for* $n>n_0$, *is also an entire function of order* ϱ *and lower type* t, *such that*

$$t \geqslant t_1^{\alpha}\, t_2^{\beta}\,.$$

Further, if T_1 *and* T_2 *are types of* $f_1(z)$ *and* $f_2(z)$ *respectively, then the type* T *of* $f(z)$ *is given by*

$$T \geqslant T_1^{\alpha}\, T_2^{\beta}\,.$$

Proof. Since $f_1(z)$ is of lower type t_1, hence

$$\frac{n}{\varrho e}\,|a_n|^{\varrho/n} > t_1-\varepsilon, \qquad n>n_1, \quad \varepsilon>0\,.$$

Or, we have

$$\left(\frac{n}{\varrho e}\right)^{\alpha}(|a_n|^{\alpha})^{\varrho/n} > (t_1-\varepsilon)^{\alpha}, \qquad n>n_1, \quad \varepsilon>0\,. \tag{2.1'}$$

Similarly for $f_2(z)$,

$$\left(\frac{n}{\varrho e}\right)^{\beta}(|b_n|^{\beta})^{\varrho/n} > (t_2-\varepsilon)^{\beta}, \qquad n>n_2, \quad \varepsilon>0\,. \tag{2.2'}$$

Since $|c_n|\sim||a_n|^{\alpha}|b_n|^{\beta}|$, we find from multiplying (2.1') and (2.2') that for large n

$$\frac{n}{\varrho e}\,|c_n|^{\varrho/n} > (t_1-\varepsilon)^{\alpha}\,(t_2-\varepsilon)^{\beta}\,.$$

Hence

$$t \geqslant t_1^{\alpha}\, t_2^{\beta}\,.$$

In the same manner the other part of the theorem follows by considering the types formulae for T_1 and T_2 respectively. The proof of $f(z)$ being an entire fuction and of order ϱ is now not difficult and so is omitted.

Corollary. *If $f_1(z)$ and $f_2(z)$ are of perfectly regular growths, so is $f(z)$ and*

$$T = T_1^{\alpha} T_2^{\beta}.$$

Theorem 2. *If $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ be two entire functions of orders ϱ_1 $(0 < \varrho_1 < \infty)$, ϱ_2 $(0 < \varrho_2 < \infty)$ and types T_1 $(0 < T_1 < \infty)$, T_2 $(0 < T_2 < \infty)$ respectively, then the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$, where $(\log | c_n |^{-1}) \sim (\log | a_n |^{-1})^{\alpha} (\log | b_n |^{-1})^{\beta}$, $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta = 1$, is also an entire function of order ϱ and type T such that*

$$T \leqslant T_1^{\alpha} T_2^{\beta},$$

provided $\varrho = \varrho_1^{\alpha} \varrho_2^{\beta}$.

Proof. Since T_1 and T_2 are respectively the types of $f_1(z)$ and $f_2(z)$, we have for every arbitrarily chosen $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, that

$$\frac{n}{e\,\varrho_1} \, | a_n |^{\varrho_1/n} < T_1 + \varepsilon_1, \qquad n > n_1,$$

$$\frac{n}{e\,\varrho_2} \, | b_n |^{\varrho_2/n} < T_2 + \varepsilon_2, \qquad n > n_2 .$$

Proceeding exactly as in Theorem 1 in obtaining the inequality (2.7), we find that, for sufficiently large n,

$$\log | c_n |^{-1} > \frac{n}{\varrho_1^{\alpha} \varrho_2^{\beta}} \left(\log \frac{n}{C}\right)^{\alpha} \left(\log \frac{n}{D}\right)^{\beta},$$

where

$$C = e\,\varrho_1 (T_1 + \varepsilon_1), \qquad D = e\,\varrho_2 (T_2 + \varepsilon_2) .$$

A similar procedure, as done after the inequality (2.7) in Theorem 1, leads to the required result.

Corollary (*of Theorems* 1 *and* 2). *We see that if* $f_1(z)$ *and* $f_2(z)$ *be each of perfectly regular growth and* $\varrho = \varrho_1^\alpha \varrho_2^\beta$, *then* $f(z)$ *is also of perfectly regular growth and*

$$T = t = t_1^\alpha t_2^\beta = T_1^\alpha T_2^\beta .$$

For, from Theorem 1

$$t \geqslant t_1^\alpha t_2^\beta ,$$

and from Theorem 2

$$T \leqslant T_1^\alpha T_2^\beta ,$$

therefore

$$t \geqslant T.$$

But $t \leqslant T$ always, and so

$$t = T .$$

3. – Theorem 3. *If* $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$, $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *be two entire functions of orders* ϱ_1 $(0 < \varrho_1 < \infty)$, ϱ_2 $(0 < \varrho_2 < \infty)$; *lower types* t_1 $(0 < t_1 < \infty)$, t_2 $(0 < t_2 < \infty)$ *respectively and* $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ *be non-decreasing functions of* n *for* $n > n_0$, *then the function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where* (i) $\log |c_n/c_{n+1}| \sim \log |a_n/a_{n+1}| + \log |b_n/b_{n+1}|$, (ii) $|c_n/c_{n+1}|$ *is again a non decreasing function of* n *for* $n > n_0$, *is also an entire function of order* ϱ *and lower type* t *such that*

$$(\varrho\, t)^{1/\varrho} \geqslant (\varrho_1\, t_1)^{1/\varrho_1} (\varrho_2\, t_2)^{1/\varrho_2}. \tag{3.1}$$

Further, if $f_1(z)$ *and* $f_2(z)$ *are of regular growths and have orders* ϱ_1, ϱ_2 *and types* T_1, T_2 *respectively* $(0 < \varrho_1 < \infty,\ 0 < \varrho_2 < \infty;\ 0 < T_1 < \infty,\ 0 < T_2 < \infty)$, *then* $f(z)$ *is also of regular growth and has order* ϱ *and type* T *such that*

$$(\varrho\, T)^{1/\varrho} \leqslant (\varrho_1\, T_1)^{1/\varrho_1} (\varrho_2\, T_2)^{1/\varrho_2} . \tag{3.2}$$

Proof. For the first part and under the hypothesis of the theorem, we have ([6], p. 75)

$$\frac{1}{\varrho} \geqslant \frac{1}{\varrho_1} + \frac{1}{\varrho_2}. \tag{3.3}$$

Now

$$\operatorname*{Lt}_{n\to\infty} \frac{\log | c_n/c_{n+1} |}{\log \{ | a_n/a_{n+1} | \, | b_n/b_{n+1} | \}} = 1 .$$

Hence, for $n \geqslant n_0$, $\varepsilon > 0$,

$$\log | c_n/c_{n+1} | < (1 + \varepsilon) \left\{ \log \left| \frac{a_n}{a_{n+1}} \right| + \log \left| \frac{b_n}{b_{n+1}} \right| \right\}. \tag{3.4}$$

Therefore, giving the values n_0, $n_0 + 1$, ..., $n - 1$ to n, we have then, from (3.4),

$$\log \left| \frac{c_{n_0}}{c_{n_0+1}} \right| < (1 + \varepsilon) \left\{ \log \left| \frac{a_{n_0}}{a_{n_0+1}} \right| + \log \left| \frac{b_{n_0}}{b_{n_0+1}} \right| \right\},$$

$$\log \left| \frac{c_{n_0+1}}{c_{n_0+2}} \right| < (1 + \varepsilon) \left\{ \log \left| \frac{a_{n_0+1}}{a_{n_0+2}} \right| + \log \left| \frac{b_{n_0+1}}{b_{n_0+2}} \right| \right\}, \text{ etc., etc.,}$$

$$\log \left| \frac{c_{n-1}}{c_n} \right| < (1 + \varepsilon) \left\{ \log \left| \frac{a_{n-1}}{a_n} \right| + \log \left| \frac{b_{n-1}}{b_n} \right| \right\}.$$

Adding these inequalities, we find that

$$\log \left| \frac{c_{n_0}}{c_n} \right| < (1 + \varepsilon) \left\{ \log \left| \frac{a_{n_0}}{a_n} \right| + \log \left| \frac{b_{n_0}}{b_n} \right| \right\}. \tag{3.5}$$

Similarly

$$\log \left| \frac{c_{n_0}}{c_n} \right| > (1 - \varepsilon) \left\{ \log \left| \frac{a_{n_0}}{a_n} \right| + \log \left| \frac{b_{n_0}}{b_n} \right| \right\}. \tag{3.6}$$

From (3.5) and (3.6), we find that

$$\log | c_n | \sim \log \{ | a_n | \, | b_n | \} . \tag{3.7}$$

Thus we conclude that condition (i) of the theorem is now reduced to (3.7), of course, with the result (3.3). (Result (3.3) can also, under the condition (3.7), be obtained from Theorem 2 of [**5**], p. 25). In other words, the first part of the theorem now reduces to a result obtained by the author (see Theorem 3 [**2**]), and therefore (3.2) is proved.

For the second part of the theorem we have ([**6**], p. 76)

$$\frac{1}{\varrho} = \frac{1}{\varrho_1} + \frac{1}{\varrho_2}$$

and this together with (3.7) roduces to a result obtained by the author (see Theorem 2 [**2**]), and thus (3.2) is proved.

4. – In this article we shall prove theorems involving certain relationships between the orders and lower orders of two or more entire functions. We begin by proving

Thorem 4. *If* $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *be two entire functions of orders* ϱ_1 *and* ϱ_2, *then the funcion* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where* $2\,(\log |c_n|^{-1})^{-1} \sim (\log |a_n|^{-1})^{-1} + (\log |b_n|^{-1})^{-1}$, *is also an entire function of order* ϱ *such that*

$$2\varrho \leqslant \varrho_1 + \varrho_2\,. \tag{4.1}$$

Further, if λ_1 *and* λ_2 *be the lower orders of* $f_1(z)$ *and* $f_2(z)$ *respectively, satisfying all the conditions as imposed upon them above, together with* $|a_n/a_{n+1}|$, *and* $|b_n/b_{n+1}|$ *forming non-decreasing functions of* n *for* $n > n_0$, *then the function* $f(z)$ *is also of lower order* λ *such that*

$$2\lambda \geqslant \lambda_1 + \lambda_2\,. \tag{4.2}$$

Corollary. *If* $f_1(z)$ *and* $f_2(z)$ *are of regular growths, then* $f(z)$ *is also of regular growth and*

$$2\varrho = \varrho_1 + \varrho_2\,. \tag{4.3}$$

The result (4.3) can also be obtained even if $f_1(z)$ and $f_2(z)$ are not of regular growths. But in that case we will have to make some other suppositions as the following theorem shows.

Theorem 5. *If* $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $f_2(z) = \sum_{n=0}^{\infty} b_n z^n$ *be two entire functions of orders* ϱ_1 $(0 < \varrho_1 < \infty)$, ϱ_2 $(0 < \varrho_2 < \infty)$; *types* T_1 $(0 < T_1 < \infty)$, T_2 $(0 < T_2 < \infty)$ *and lower types* t_1 $(0 < t_1 < \infty)$, t_2 $(0 < t_2 < \infty)$ *and if* $| a_n/a_{n+1} |$, $| b_n/b_{n+1} |$ *be non-decreasing functions of* n *for* $n > n_0$, *then* $f(z) = \sum_{n=0}^{\infty} c_n z^n$, *where* $2(\log | c_n |^{-1})^{-1} \sim (\log | a_n |^{-1})^{-1} + (\log | b_n |^{-1})^{-1}$ *and* $| c_n/c_{n+1} |$ *is a non-decreasing function of* n *for* $n > n_0$, *is also an entire function of order* ϱ *such that*

$$2\,\varrho = \varrho_1 + \varrho_2 \,.$$

Proof of Theorem 4. First we show that $f(z)$ is an entire function. As $f_1(z)$ is an entire function, hence

$$(\log | a_n |^{-1})^{-1} < (\log (R - \varepsilon)^n)^{-1}, \qquad n > n_1, \quad \varepsilon > 0 \,.$$

Similarly for $f_2(z)$, we have

$$(\log | b_n |^{-1})^{-1} < (\log (R - \varepsilon)^n)^{-1}, \qquad n > n_2, \quad \varepsilon > 0 \,.$$

Since

$$2\,(\log | a_n |^{-1})^{-1} \sim (\log | a_n |^{-1})^{-1} + (\log | b_n |^{-1})^{-1}, \tag{4.4}$$

hence, for sufficiently large n,

$$2\,(\log | c_n |^{-1})^{-1} < 2\,(\log (R - \varepsilon)^n)^{-1},$$

and so $f(z)$ is an entire function.

Again, $f_1(z)$ and $f_2(z)$ are of orders ϱ_1 and ϱ_2 respectively, therefore

$$\frac{n \log n}{\log | a_n |^{-1}} < (\varrho_1 + \varepsilon), \qquad n > n_1, \quad \varepsilon > 0, \tag{4.5}$$

$$\frac{n \log n}{\log | b_n |^{-1}} < (\varrho_2 + \varepsilon), \qquad n > n_2, \quad \varepsilon > 0. \tag{4.6}$$

Using (4.4), we find from (4.5) and (4.6) by adding them, that for large n

$$\frac{2n \log n}{\log | c_n |^{-1}} < \varrho_1 + \varrho_2 + 2\varepsilon,$$

and (4.1) follows. Further, since $|a_n/a_{n+1}|$ is non-decreasing, we find on using (1.1) in case of $f_1(z)$,

$$\underset{n\to\infty}{\mathrm{Lt}} \frac{n \log n}{\log |a_n|^{-1}} = \lambda_1 .$$

So that

$$\frac{n \log n}{\log |a_n|^{-1}} > \lambda_1 - \varepsilon, \qquad n > n_1, \quad \varepsilon > 0 .$$

Similarly for $f_2(z)$, we have

$$\frac{n \log n}{\log |b_n|^{-1}} > \lambda_2 - \varepsilon, \qquad n > n_2, \quad \varepsilon > 0 .$$

Proceeding as above for getting the result (4.1), we easily see that (4.2) is proved.

Proof of Theorem 5. Since $f_1(z)$ is of type T_1 $(0 < T_1 < \infty)$, hence using (1.2), we see that

$$\frac{n \log n}{\log |a_n|^{-1}} < \frac{n}{\log |a_n|^{-1}} \log \left(e\, \varrho_1 (T_1 + \varepsilon)\right) + \varrho_1, \quad n > n_1, \quad \varepsilon > 0$$

$$\sim \varrho_1 + o(1) .$$

Similarly for large n, we have for $f_2(z)$ the inequality

$$\frac{n \log n}{\log |b_n|^{-1}} < \varrho_2 + o(1) .$$

Hence from (4.4),

$$\frac{2n \log n}{\log |c_n|^{-1}} < \varrho_1 + \varrho_2 + o(1)$$

for large n. Therefore

$$2\, \varrho \leqslant \varrho_1 + \varrho_2 . \tag{4.7}$$

Again, $|a_n/a_{n+1}|$ is non-decreasing and $f_1(z)$ is of lower type t_1 $(0 < t_1 < \infty)$, therefore on using (1.2) we have

$$\frac{n \log n}{\log |a_n|^{-1}} > \frac{n}{\log |a_n|^{-1}} \log ((t_1 - \varepsilon)\, e\, \varrho_1) + \varrho_1, \qquad n > n_1, \quad \varepsilon > 0 .$$

Similarly for $f_2(z)$ we have

$$\frac{n \log n}{\log |b_n|^{-1}} > \frac{n}{\log |b_n|^{-1}} \log ((t_2 - \varepsilon)\, e\, \varrho_2) + \varrho_2, \qquad n > n_2, \quad \varepsilon > 0 .$$

Hence

$$2\,\varrho \geqslant 2\,\lambda = \mathop{\mathrm{Lt}}_{n\to\infty} \frac{2n \log n}{\log |c_n|^{-1}} \geqslant \varrho_1 + \varrho_2 , \tag{4.8}$$

where λ is the lower order of $f(z)$. From (4.7) and (4.8) we get

$$\varrho = \varrho_1 + \varrho_2 .$$

Finally, I wish to convey my best thanks to Dr. S. C. Mitra for his helpful guidance.

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