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On a Relation Between Harmonic Summability and Lebesgue Summability. (**)

1. – An infinite series $\sum_1^\infty a_n$ is defined to be LEBESGUE summable (shortly L-summable) to the sum s, if the sine series

(1.1) $$\sum_1^\infty a_n \frac{\sin nt}{n} = F(t)$$

is convergent in some interval $-\tau < t < \tau$, and if

(1.2) $$t^{-1} F(t) \to s \qquad \text{as} \qquad t \to 0 .$$

The method is not regular ([1], p. 89). Let us write

(1.3) $$A_n^k = \binom{n+k}{n}, \qquad \text{and} \qquad s_n^k = \sum_1^n A_{n-\nu}^k a_\nu .$$

The series $\sum_1^\infty a_n$ is said to be summable (C, k) for some $k \geqslant 0$ to the sum s, if

(1.4) $$\lim_{n\to\infty} \frac{s_n^k}{A_n^k} = s \qquad\qquad ([1], \text{ p. } 97) .$$

The series $\sum_1^\infty a_n$ with partial sums $s_n = a_1 + a_2 + \ldots + a_n$ is said to be harm-

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onic summable [3] to the sum s, if $t_n \to s$ as $n \to \infty$, where

$$(1.5) \qquad t_n = \frac{S_n}{\log n} = \frac{s_1\, p_{n-1} + s_2\, p_{n-2} + \dots + s_n\, p_0}{\log n}, \qquad p_n = \frac{1}{n+1}.$$

We write $P_n = p_0 + p_1 + \dots + p_n$, so that $P_n \sim \log n$.

Otto Szász [4] has proved that if $\sum_1^\infty a_n$ is summable (C, $1-\alpha$) for some positive $\alpha < 1$, and if

$$\sum_1^n \left| s_\nu^{1-\alpha} - s_{\nu-1}^{1-\alpha} \right| = O(n^{1-\alpha}) \quad \text{as} \quad n \to \infty,$$

then the series $\sum_1^\infty a_n$ is L-summable.

The object of this paper is to prove an analogous theorem for harmonic summability. We prove

Theorem. *If* $\sum_1^\infty a_n$ *is harmonic summable and if*

$$(1.6) \qquad \sigma_n = \sum_1^n \left| S_\nu - S_{\nu-1} \right| = O(\log n) \quad \textit{as} \quad n \to \infty,$$

then the series $\sum_1^\infty a_n$ *is* L-*summable.*

We may assume, without any loss of generality, that $S_n = o(\log n)$ as $n \to \infty$. Let $W_n = \sum_{\nu=n}^\infty \frac{b_\nu}{\nu}$, where $b_\nu = S_\nu - S_{\nu-1}$. By Abel's transformation,

$$W_n = \sum_{\nu=n}^\infty \frac{b_n + b_{n+1} + \dots + b_\nu}{\nu(\nu+1)}$$

$$= \sum_{\nu=n}^\infty \frac{S_\nu - S_{n-1}}{\nu(\nu+1)} = o\left[\sum_{\nu=n}^\infty \frac{\log \nu}{\nu(\nu+1)}\right],$$

hence

$$(1.7) \qquad W_n = o\big((\log n)/n\big).$$

This gives

$$(1.8) \qquad \sum_1^n W_\nu = n\, W_{n+1} + S_n = o(\log n).$$

2. – We set

$$(p_0 + p_1 x + \ldots + p_n x^n + \ldots)^{-1} \sim c_0 + c_1 x + \ldots + c_n x^n + \ldots . \tag{2.1}$$

We have from (1.5)

$$\sum_{n=1}^{\infty} S_n x^{n-1} \sim \sum_{1}^{\infty} a_n x^{n-1} \sum_{n=0}^{\infty} P_n x^n \sim \Big(\sum_{1}^{\infty} a_n x^{n-1}\Big)(1-x)^{-1}\Big(\sum_{0}^{\infty} p_n x^n\Big) \tag{2.2}$$

and therefore

$$\sum_{n=1}^{\infty} a_n x^{n-1} \sim (1-x)\Big(\sum_{n=1}^{\infty} S_n x^{n-1}\Big)(c_0 + c_1 x + \ldots + c_n x^n + \ldots) , \tag{2.3}$$

from which we obtain

$$a_n = c_0 (S_n - S_{n-1}) + c_1 (S_{n-1} - S_{n-2}) + \ldots + c_{n-2} (S_2 - S_1) + c_{n-1} S_1 . \tag{2.4}$$

It is known ([2], [5], [6]) that

$$c_n = O\big(1/(n \log^2 n)\big) , \tag{2.5}$$

$$d_n = c_0 + c_1 + \ldots + c_n = O(1/\log n) . \tag{2.6}$$

We require the following lemma for proving our theorem.

Lemma. *The series*

$$\sum_{n=0}^{\infty} c_n \frac{\sin (n+\nu)t}{(n+\nu)t} = T_\nu(t) \tag{2.7}$$

is absolutely convergent, and

$$T_\nu(t) = O\big(1/(\nu\, t \log \tau)\big), \qquad \tau = [1/t], \quad 0 < t < 1, \quad \nu \geqslant 1, \tag{2.8}$$

and

$$\sum_{n=0}^{\infty} c_n \frac{1 - \cos (n+\nu-1)t}{n+\nu} = O\left[\frac{t}{\log \tau}\right], \qquad 0 < t < 1, \quad \nu \geqslant 1. \tag{2.9}$$

Absolute convergence of the series (2.7) follows from (2.5). To prove (2.8) and (2.9) we note that

$$\Delta \frac{\sin (n+\nu)t}{(n+\nu)t} = \frac{\sin (n+\nu)t}{(n+\nu)t} - \frac{\sin (n+\nu+1)t}{(n+\nu+1)t}$$

$$= \frac{\sin (n+\nu)t - (n+\nu)\{\sin (n+\nu+1)t - \sin (n+\nu)t\}}{(n+\nu)(n+\nu+1)t},$$

hence

$$\Delta \frac{\sin (n+\nu)t}{(n+\nu)t} = O(1/\nu), \tag{2.10}$$

and

$$\Delta \frac{1-\cos (n+\nu-1)t}{n+\nu} = \frac{1-\cos (n+\nu-1)t}{n+\nu} - \frac{1-\cos (n+\nu)t}{n+\nu+1}$$

$$= \frac{1-\cos (n+\nu-1)t + (n+\nu)\{\cos (n+\nu)t - \cos (n+\nu-1)t\}}{(n+\nu)(n+\nu+1)}$$

$$= \frac{2\sin^2 \big((n+\nu-1)/2\big)t - 2(n+\nu)\sin (t/2) \sin \big(n+\nu-(1/2)\big)t}{(n+\nu)(n+\nu+1)},$$

hence

$$\Delta \frac{1-\cos (n+\nu-1)t}{n+\nu} = O(t^2)\,. \tag{2.11}$$

Now

$$T_\nu(t) = \sum_{n=0}^{\infty} c_n \frac{\sin (n+\nu)t}{(n+\nu)t} = \sum_{n=0}^{\tau} \cdots + \sum_{\tau+1}^{\infty} \cdots = \Sigma_1 + \Sigma_2\,,$$

say. Using (2.6) and (2.10) we have

$$\Sigma_1 = \sum_{n=0}^{\tau} c_n \frac{\sin (n+\nu)t}{(n+\nu)t}$$

$$= \sum_{n=0}^{\tau-1} d_n \Delta \frac{\sin (n+\nu)t}{(n+\nu)t} + d_\tau \frac{\sin (\tau+\nu)t}{(\tau+\nu)t} = O\left[\frac{1}{\nu\, t \log \tau}\right],$$

and

$$\Sigma_2 = \sum_{\tau+1}^{\infty} c_n \frac{\sin (n+\nu)t}{(n+\nu)t} = O\left[\sum_{\tau+1}^{\infty} \frac{1}{n \log^2 n} \frac{1}{\nu\, t}\right] = O\left[\frac{1}{\nu\, t \log \tau}\right].$$

This proves (2.8). The proof of (2.9) is similar except that we use (2.11) in place of (2.10).

3. – Proof of the theorem. Employing (1.1) and (2.4) we have

$$\frac{F(t)}{t} = \sum_{1}^{\infty} n^{-1} a_n \sin nt = \sum_{n=1}^{\infty} n^{-1} \sin nt \sum_{\nu=1}^{n} c_{n-\nu} (S_\nu - S_{\nu-1}),$$

hence

$$\frac{F(t)}{t} = \sum_{\nu=1}^{\infty} (S_\nu - S_{\nu-1}) \sum_{n=\nu}^{\infty} n^{-1} c_{n-\nu} \sin nt, \tag{3.1}$$

the interchange of summation is justified, as the double series is absolutely convergent. In fact from (2.5)

$$\left| \sum_{n=\nu}^{\infty} n^{-1} c_{n-\nu} \sin nt \right| \leqslant \frac{1}{\nu} \sum_{n=0}^{\infty} | c_n |$$

$$= O\left[\frac{1}{\nu} \sum \frac{1}{n \log^2 n}\right] = O(1/\nu).$$

It remains to prove the convergence of

$$\sum_{1}^{\infty} \nu^{-1} | S_\nu - S_{\nu-1} |.$$

Now employing (1.6) we have

$$\sum_{1}^{n} \nu^{-1} | S_\nu - S_{\nu-1} | = n^{-1} \sigma_n + \sum_{1}^{n-1} \sigma_\nu \frac{1}{\nu(\nu+1)}$$

$$= O\left[\frac{\log n}{n}\right] + O\left[\sum_{1}^{n-1} \frac{\log \nu}{\nu(\nu+1)}\right] = O(1).$$

Thus all series in (3.1) are absolutely convergent, and

$$t^{-1} F(t) = \sum_{1}^{\infty} (S_\nu - S_{\nu-1}) T_\nu(t).$$

Now, choose a positive number μ and put $n = [\mu t^{-1}]$,

$$\sum_{1}^{\infty} (S_\nu - S_{\nu-1}) T_\nu(t) = \left(\sum_{1}^{n} + \sum_{n+1}^{\infty}\right)(S_\nu - S_{\nu-1}) T_\nu(t),$$

hence

$$\sum_{1}^{\infty} (S_\nu - S_{\nu-1})\, T_\nu(t) = \sum\nolimits_1 + \sum\nolimits_2 , \tag{3.2}$$

say. From (2.8) we have

$$\sum\nolimits_2 = O\left[\frac{1}{t \log \tau}\right] \sum_{n+1}^{\infty} (1/\nu)\, |S_\nu - S_{\nu-1}|$$

and from (1.6)

$$\sum_{n+1}^{\infty} \frac{1}{\nu} |S_\nu - S_{\nu-1}| = \frac{-\sigma_n}{n+1} + \sum_{n+1}^{\infty} \sigma_\nu \frac{1}{\nu(\nu+1)}$$

$$= O\left[\frac{\log n}{n}\right] + O\left[\sum_{n+1}^{\infty} \frac{\log \nu}{\nu(\nu+1)}\right] = O\left[\frac{\log n}{n}\right].$$

Hence

$$\sum\nolimits_2 = O\left[\frac{\log n}{n\, t \log \tau}\right] = \frac{\log \mu}{\mu} \cdot O(1) . \tag{3.3}$$

Furthermore

$$\sum\nolimits_1 = \sum_{1}^{n} (S_\nu - S_{\nu-1})\, T_\nu(t) = \sum_{1}^{n} (W_\nu - W_{\nu+1})\, \nu\, T_\nu(t)$$

$$= \sum_{1}^{n} W_\nu \left\{ \nu\, T_\nu(t) - (\nu - 1)\, T_{\nu-1}(t) \right\} - n\, W_{n+1}\, T_n(t)$$

$$= \frac{\sin t}{t} \sum_{1}^{n} W_\nu \sum_{0}^{\tau} c_\mu - \frac{1}{t} \sum_{1}^{n} W_\nu \sum_{0}^{\tau} \frac{\mu\, c_\mu}{\mu + \nu} \left[\sin t - \frac{\sin (\mu + \nu - 1)t}{\mu + \nu - 1}\right] +$$

$$+ \frac{1}{t} \sum_{1}^{n} \nu\, W_\nu \sum_{\tau+1}^{\infty} \frac{c_\mu}{\mu + \nu} \left[\sin t - \frac{\sin (\mu + \nu - 1)t}{\mu + \nu - 1}\right] +$$

$$+ \frac{1}{t} \sum_{1}^{n} W_\nu \sum_{\tau+1}^{\infty} c_\mu \frac{\sin (\mu + \nu - 1)t}{\mu + \nu - 1} - \frac{1 - \cos t}{t} \sum_{1}^{n} \nu\, W_\nu \sum_{0}^{\infty} c_\mu \frac{\sin (\mu + \nu - 1)t}{\mu + \nu} -$$

$$- \frac{\sin t}{t} \sum_{1}^{n} \nu\, W_\nu \sum_{0}^{\infty} c_\mu \frac{1 - \cos (\mu + \nu - 1)\, t}{\mu + \nu} - n\, W_{n+1}\, T_n(t),$$

hence

(3.4) $$\sum_1 = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + \varphi_5 + \varphi_6 + \varphi_7\,, \qquad \text{say.}$$

Using (1.8) and (2.6) we have

(3.5) $$\varphi_1 = \frac{\sin t}{t}\sum_1^n W_\nu \sum_0^\tau c_\mu = \frac{\sin t}{t} d_\tau \sum_1^n W_\nu = o\left[\frac{\log n}{\log \tau}\right] = \log \mu \cdot o(1)\,.$$

Since

$$\sin t - \frac{\sin(\mu+\nu-1)t}{\mu+\nu-1} = O((\mu+\nu)t^2)\,,$$

we have by (1.7) and (2.5)

$$\varphi_2 = -\frac{1}{t}\sum_1^n W_\nu \sum_0^\tau \frac{\mu\, c_\mu}{\mu+\nu}\left[\sin t - \frac{\sin(\mu+\nu-1)t}{\mu+\nu-1}\right] =$$

$$= O\left[t\sum_1^n |W_\nu| \sum^\tau \frac{1}{\log^2 \mu}\right] = o\left[t \log^2 n \cdot \frac{\tau}{\log^2 \tau}\right],$$

hence

(3.6) $$\varphi_2 = \log^2 \mu \cdot o(1)\,,$$

and since

$$\text{sint}\, t - \frac{\sin(\mu+\nu-1)t}{\mu+\nu-1} = O(t),$$

we have

$$\varphi_3 = \frac{1}{t}\sum_1^n \nu\, W_\nu \sum_{\tau+1}^\infty \frac{c_\mu}{\mu+\nu}\left[\sin t - \frac{\sin(\mu+\nu-1)t}{\mu+\nu-1}\right]$$

$$= O\left[\sum_1^n \nu\, |W_\nu| \sum_{\tau+1}^\infty \frac{1}{\mu^2 \log^2 \mu}\right] = o\left[n \log n \cdot \frac{1}{\tau \log^2 \tau}\right],$$

hence

(3.7) $$\varphi_3 = \mu \log \mu \cdot o(1)\,.$$

Since $\left|\sin(\mu+\nu-1)t\right| \leqslant 1$, we have by (1.7) and (2.5)

$$\varphi_4 = \frac{1}{t}\sum_{1}^{n} W_\nu \sum_{\tau+1}^{\infty} c_\mu \frac{\sin(\mu+\nu-1)t}{\mu+\nu-1}$$

$$= O\left[\frac{1}{t}\sum_{1}^{n} \left|W_\nu\right| \sum_{\tau+1}^{\infty} \frac{|c_\mu|}{\mu}\right]$$

$$= o\left[\frac{1}{t}\sum_{1}^{n}\frac{\log n}{n}\sum_{\tau+1}^{\infty}\frac{1}{\mu^2\log^2\mu}\right] = o\left[\frac{1}{t\,\tau}\cdot\frac{\log^2 n}{\log^2\tau}\right],$$

hence

$$\varphi_4 = \log^2\mu \cdot o(1)\,, \tag{3.8}$$

and

$$\varphi_5 = -\frac{1-\cos t}{t}\sum_{1}^{n}\nu\, W_\nu \sum_{0}^{\infty} c_\mu \frac{\sin(\mu+\nu-1)t}{\mu+\nu}$$

$$= O\left[\frac{t^2}{t}\sum_{1}^{n}\left|W_\nu\right|\left|\sum_{0}^{\infty}\right| c_\mu\left|\right|\right] = o\left[t\sum_{1}^{n}\frac{\log n}{n}\right] = o(t\log^2 n)\,,$$

hence

$$\varphi_5 = \log^2\mu \cdot o(1)\,. \tag{3.9}$$

Using (1.7) and (2.9), we have

$$\varphi_6 = -\frac{\sin t}{t}\sum_{1}^{n}\nu\, W_\nu\sum_{0}^{\infty} c_\mu \frac{1-\cos(\mu+\nu-1)t}{\mu+\nu}$$

$$= o\left[\sum_{1}^{n}(\log\nu)\frac{t}{\log\tau}\right] = o\left[\frac{t\,n\log n}{\log\tau}\right],$$

hence

$$\varphi_6 = \mu\log\mu \cdot o(1)\,. \tag{3.10}$$

Finally using (1.7) and (2.8) we obtain

$$\varphi_7 = -\,n\, W_{n+1}\, T_n(t) = o(\log n)\cdot O\big(1/(n\, t\log\tau)\big)\,.$$

hence

$$\varphi_7 = \frac{\log \mu}{\mu} \cdot o(1) . \tag{3.11}$$

Collecting (3.4) through (3.11) we obtain

$$\textstyle\sum_1 = \mu \log \mu \cdot o(1) , \tag{3.12}$$

which together with (3.2) and (3.3) gives

$$t^{-1} F(t) = \frac{\log \mu}{\mu} \cdot O(1) + \mu \log \mu \cdot o(1) \qquad \text{as} \quad t \to 0.$$

Consequently

$$\limsup_{t \to 0} t^{-1} \left| F(t) \right| \leqslant \frac{\log \mu}{\mu} \cdot O(1) ,$$

μ being arbitrarily large and $O(1)$ indipendent of μ, we get finally

$$t^{-1} F(t) \to 0 \qquad \text{as} \qquad t \to 0 .$$

This completes the proof of our theorem.

References.

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Summary.

Otto Szász has obtained a set of sufficient conditions under which an infinite series which is summable (C, $1 - \alpha$), $0 < \alpha < 1$, *is also* L-*summable. Here the author obtains an analogous relation between harmonic summability and Lebesgue summability.*

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