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The Phragmén-Lindelöf Principle for Functions of Several Complex Variables. (**)

1. – Let $\mathscr{C}$ denote the space of complex numbers, and let

$$\mathscr{C}^n = \{ z = (z_1, \ldots, z_n) : \ z_j \in \mathscr{C} \} \qquad (j = 1, \ldots, n; \quad n \geqslant 2)$$

be the space of ordered n-tuples of complex numbers. The usual (product) topology on $\mathscr{C}^n$ has a base consisting of all *open polydiscs* of the form

$$P = \{ z \in \mathscr{C}^n : \ | z_j - a_j | < r_j, \ (j = 1, \ldots, n) \}.$$

For any set $A \subset \mathscr{C}^n$, let A^i, A^c and A^b denote respectively the interior, closure and boundary of A, and let $A_j = p_j[A]$ $(j = 1, \ldots, n)$ be the images of A under continuous projections into each coordinate space $\mathscr{C}$. Then, in general, $A \subset A_1 \times \times \ldots \times A_n$ and in case $A = A_1 \times \ldots \times A_n$ (a set of the product form), we have $A^i = A_1^i \times \ldots \times A_n^i$ and $A^c = A_1^c \times \ldots \times A_n^c$, and we define the edge of A by $A^e = A_1^b \times \ldots \times A_n^b$, which is, in fact a subset of A^b.

Let $f(z) = f(z_1, \ldots, z_n)$ be a function of the n complex variables z_j, defined on an open set $\mathscr{U} \subset \mathscr{C}^n$. Then [1, p. 2, Cor. 3] $f(z)$ is analytic in $\mathscr{U}$, if and

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only if, for every point $z^0 = (z_1^0, \ldots, z_n^0) \in \mathscr{U}$, the functions

$$F(z_j) = f(z_1^0, \ldots, z_{j-1}^0, z_j, z_{j+1}^0, \ldots, z_n^0) \qquad (j = 1, \ldots, n) \tag{1.1}$$

of one complex variable z_j are analytic respectively in the plane open sets

$$\mathscr{U}_j = \left\{ z_j \in \mathscr{C}\colon\ (z_1^0, \ldots, z_{j-1}^0, z_j, z_{j+1}^0, \ldots, z_n^0) \in \mathscr{U} \right\} \qquad (j = 1, \ldots, n). \tag{1.2}$$

The object of this paper to formulate and prove the PHRAGMÉN-LINDELÖF extension of the maximum-modulus principle for functions of several complex variables, analytic in an *open subset* of $\mathscr{C}^n$.

2. – We shall first give a proof of the maximum-modulus principle, which may be stated as

Let $f(z) = f(z_1, \ldots, z_n)$ be analytic in the closure of a bounded open set $\mathscr{U} \subset \mathscr{C}^n$ and $|f(z)| \leqslant M$ on $\mathscr{U}^b$. Then $|f(z)| \leqslant M$ in $\mathscr{U}$. Moreover, $|f(z)| = M$ at a point in $\mathscr{U}$ only if $f(z)$ is constant in the component (maximal connected subset) of $\mathscr{U}$ containing that point.

Proof. Let $z^0 = (z_1^0, \ldots, z_n^0)$ be an arbitrary fixed point in $\mathscr{U}$ and $\mathscr{U}_j$ be defined as in (1.2). Then it is easy to see that

$$\prod_{k=1}^{j-1} \{z_k^0\} \times \mathscr{U}_j^c \times \prod_{k=j+1}^{n} \{z_k^0\} \subset \mathscr{U}^c$$

and

$$\prod_{k=1}^{j-1} \{z_k^0\} \times \mathscr{U}_j^b \times \prod_{k=j+1}^{n} \{z_k^0\} \subset \mathscr{U}^b \qquad (j = 1, \ldots, n).$$

Let $F(z_j)$ be defined as in (1.1). Then, $F(z_j)$ being analytic respectively in $\mathscr{U}_j^c$ and

$$|F(z_j)| \leqslant M \quad \text{on} \quad \mathscr{U}_j^b \qquad (j = 1, \ldots, n)$$

we have, by the maximum-modulus principle for functions of one complex variable,

$$|F(z_j)| \leqslant M \quad \text{in} \quad \mathscr{U}_j \qquad (j = 1, \ldots, n).$$

Since $z_j^0 \in \mathscr{U}_j$, therefore, $|f(z^0)| \leqslant M$. Hence, z^0 being any point in $\mathscr{U}$, it follows that $|f(z)| \leqslant M$ in $\mathscr{U}$.

Now, let z^0 be a point in $\mathscr{U}$ such that $|f(z^0)| = M$ and C be the component of $\mathscr{U}$ containing z^0, and define

$$C_j = \left\{ z_j \in \mathscr{C}\colon (z_1^0, \ldots, z_{j-1}^0, z_j, z_{j+1}^0, \ldots, z_n^0) \in C \right\} \qquad (j = 1, \ldots, n),$$

which are, in fact, the components of $\mathcal{U}_j$ containing z_0^j. Then, since $|F(z_j^0)| = M$, we have

$$F(z_j) = F(z_j^0) = f(z^0) \quad \text{in} \quad C_j \qquad (j = 1, \ldots, n)$$

and therefore, by [1, p. 2, Cor. 3],

$$f(z) = f(z^0) \quad \text{in} \quad C_1 \times \ldots \times C_n .$$

Hence, as $C \subset C_1 \times \ldots \times C_n$, it follows that $f(z)$ is constant in C.

Remark. In case $\mathcal{U}$ is an open set of the product form, we can replace the boundary $\mathcal{U}^b$ by the edge $\mathcal{U}^e \subset \mathcal{U}^b$. Also if $\mathcal{U}$ is a domain (connected open set), then $C = \mathcal{U}$.

Theorem 1. *Let $f(z) = f(z_1, \ldots, z_n)$ be analytic in the closure of a bounded open set $\mathcal{U} \subset \mathcal{C}^n$ except at a finite number of points*

$$a^{(k)} = (a_1^{(k)}, \ldots, a_n^{(k)}) \in \mathcal{U}^b \qquad (k = 1, \ldots, m)$$

and

(2.1) $$|f(z)| \leqslant M \quad on \quad \mathcal{U}^b \sim A, \qquad A = \{a^{(k)}: k = 1, \ldots, m\}.$$

Further, let there be another function $w(z) = w(z_1, \ldots, z_n)$ analytic in $\mathcal{U}^c \sim A$ and satisfying the conditions:

(i) $w(z) \neq 0$ *in* $\mathcal{U}$,

(ii) $|w(z)| \leqslant 1$ *on* $\mathcal{U}^b \sim A$ *and*

(iii) *given $\varepsilon > 0$, we can find a system of m domains, each containing a point $a^{(k)}$ and having an arbitrarily small diameter such that, on part in $\mathcal{U}$ of the boundary of their union*

$$|f(z) \cdot \{w(z)\}^{\varepsilon}| \leqslant M.$$

Then

$$|f(z)| \leqslant M \qquad in \ \mathcal{U} .$$

Proof. Given $\varepsilon > 0$, consider the function $f_\varepsilon(z) = f(z)\{w(z)\}^\varepsilon$ which is analytic in $\mathcal{U}^c \sim A$. Let $z^0 = (z_1^0, \ldots, z_n^0)$ be an arbitrary fixed point in $\mathcal{U}$. Then, by (iii), we can find domains $D_{\varepsilon,k}$ $(k = 1, \ldots, m)$ such that

$$a^{(k)} \in D_{\varepsilon,k}, \qquad z^0 \notin D_{\varepsilon,k} \qquad (k = 1, \ldots, m)$$

and

(2.2) $$|f_\varepsilon(z)| \leqslant M \quad \text{on} \quad \mathcal{U} \cap D_\varepsilon^b, \qquad D_\varepsilon = \bigcup_{k=1}^{m} D_{\varepsilon,k} .$$

Since $A \subset D_\varepsilon$, therefore, $f_\varepsilon(z)$ is analytic in $\mathscr{U}^c \sim D_\varepsilon$ and also, by (ii) and (2.1), we have

$$|f_\varepsilon(z)| \leqslant M \quad \text{on} \quad \mathscr{U}^b \sim D_\varepsilon. \tag{2.3}$$

Now, let $\mathscr{U}_\varepsilon = \mathscr{U} \sim D_\varepsilon^c$, which again is a bounded open set in $\mathscr{C}^n$. Then it is easy to see that

$$\mathscr{U}_\varepsilon^c = \mathscr{U}^c \sim D_\varepsilon \quad \text{and} \quad \mathscr{U}_\varepsilon^b = (\mathscr{U} \cap D_\varepsilon^b) \cup (\mathscr{U}^b \sim D_\varepsilon).$$

Thus, $f_\varepsilon(z)$ being analytic in $\mathscr{U}_\varepsilon^c$ and, by (2.2) and (2.3),

$$|f_\varepsilon(z)| \leqslant M \quad \text{on} \quad \mathscr{U}_\varepsilon^b$$

we have, by the maximum modulus principle,

$$|f_\varepsilon(z)| \leqslant M \quad \text{in} \quad \mathscr{U}_\varepsilon.$$

Since $z^0 \in \mathscr{U}_\varepsilon$, therefore

$$|f(z^0) \cdot \{w(z^0)\}^\varepsilon| \leqslant M,$$

or, by (i),

$$|f(z^0)| \leqslant M\,|w(z^0)|^{-\varepsilon}.$$

This being true for all $\varepsilon > 0$ and $f(z^0)$ being independent of ε, we have, on making $\varepsilon \to 0$, $|f(z^0)| \leqslant M$. Hence, z^0 being any point in $\mathscr{U}$, the result follows.

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Reference.

[1] M. Hervé, *Several Complex Variables*, Oxford University Press, 1963.

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