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Motion of a Finite Sphere in Rotating Liquid: Motion Started Impulsively from Rest. (**)

1. - Introduction.

HOWARTH ([1], 1951) has investigated the problem of the flow engendered by a sphere rotating uniformly about a diameter in otherwise undisturbed fluid. He considers the sphere to be made up of two hemispheres joined smoothly at the equator. The flow being symmetrical about the equatorial plane, the boundary layers originate at the poles and develop towards the equator where they impinge on each other. He has obtained a solution of the boundary layer equations in the form of power series and has shown that the flow at the poles approximates to the rotating disc solution of VON KÁRMÁN (1921). Due to lack of rapid convergence this solution cannot be used in the vicinity of the equator. Even the approximate solution of the same problem by the KÁRMÁN-Momentum Integral Method does not give satisfactory results near the equator, because some of the assumptions involved break down. The solutions are defective in so far as they do not give any indication of outflow of the liquid near the equator. HOWARTH attributes the cause of the failure of his solutions near the equator to the boundary layer equations, which, he says must fail to represent the region of interaction between the two impinging layers on account of the parabolic character of equations.

NIGAM ([2], 1954) does not agree with this explanation. In reference [2] he has shown that it is possible to construct a solution in power series forms which fulfils all the physical requirements of the problem. These solutions

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have a definite advantage over those of HOWARTH, because apart from the inflow at the poles they give a clear indication of the outflow near the equator.

In another paper NIGAM and RANGASAMI ([3], 1953) have discussed the growth of motion in the earlier stages of development, caused by a sphere which at the time $t = 0$ is suddenly made to rotate with a constant angular spin about a diameter in fluid otherwise undisturbed.

In the present Note we have discussed the growth of motion, in the earlier stages of its development, caused by a sphere which at the time $t = 0$ is suddenly made to rotate with a constant angular spin $\sigma \Omega$ about a diameter in the liquid rotating about the same diameter with an angular velocity Ω. The solutions have a limitation in that they give initial motion only. They give no information regarding the time after which the steady state is established.

2. - Equations of motion.

In curvilinear coordinates the equations of motion (HOWARTH [4]) are

$$\frac{\partial u}{\partial t} + \frac{u}{h_1}\frac{\partial u}{\partial \xi} + \frac{v}{h_2}\frac{\partial u}{\partial \eta} + w\frac{\partial u}{\partial \zeta} - K_2 u v + K_1 v^2 = -\frac{1}{\varrho h_1}\frac{\partial p}{\partial \xi} + \nu \frac{\partial^2 u}{\partial \zeta^2}, \tag{1}$$

$$\frac{\partial v}{\partial t} + \frac{u}{h_1}\frac{\partial v}{\partial \xi} + \frac{v}{h_2}\frac{\partial v}{\partial \eta} + w\frac{\partial v}{\partial \zeta} + K_2 u^2 - K_1 u v = -\frac{1}{\varrho h_2}\frac{\partial p}{\partial \eta} + \nu \frac{\partial^2 v}{\partial \zeta^2} \tag{2}$$

and the equation of continuity is

$$\frac{1}{h_1}\frac{\partial u}{\partial \xi} + \frac{1}{h_2}\frac{\partial v}{\partial \eta} + \frac{\partial w}{\partial \zeta} - K_1 u - K_2 v = 0 , \tag{3}$$

where h_1, h_2, K_1, K_2 have the meanings as explained in the above reference.

In this problem we use spherical polar coordinates r, θ, φ with r measured radially outwards from the centre of the sphere, θ measured from the axis of the rotation and φ the azimuth. In order to preserve w for the velocity normal to the surface (i.e. in the direction r increasing), we shall use u, v for the velocities in the directions in θ, φ increasing respectively. Then writing

$$h_1 = r, \qquad h_2 = 0, \qquad K_1 = 0, \qquad K_2 = -(1/r)\cot\theta$$

as in reference [1] and neglecting the terms of the azimuthal variation, equations (1), (2) and (3) become

$$\frac{\partial u}{\partial t} + \frac{u}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial r} - \frac{v^2}{r}\cot\theta = -\frac{1}{\varrho}\frac{\partial p}{r\,\partial \theta} + \nu\frac{\partial^2 u}{\partial r^2}, \tag{4}$$

$$\frac{\partial v}{\partial t} + \frac{u}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial r} + \frac{uv}{r}\cot\theta = \nu\frac{\partial^2 v}{\partial r^2}, \tag{5}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial r} + \frac{u}{r}\cot\theta = 0\,. \tag{6}$$

The pressure gradient along the surface can be computed for the frictionless flow at a large distance from the sphere and for it we write

$$\frac{1}{\varrho}\frac{\partial p}{r\,\partial \theta} = v\frac{\partial v}{\partial s} = v\frac{\partial v}{r\,\partial \theta} = r\sin\theta\cdot\Omega\,\frac{1}{r}\,r\cos\theta\cdot\Omega = r\sin\theta\cos\theta\cdot\Omega^2$$

as $v = r\sin\theta\cdot\Omega$.

Hence equation (4) becomes

$$\frac{\partial u}{\partial t} + \frac{u}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial r} - \frac{v^2}{r}\cot\theta = -r\sin\theta\cos\theta\cdot\Omega^2 + \nu\frac{\partial^2 u}{\partial r^2}. \tag{7}$$

If the liquid is rotating with an angular velocity Ω and the sphere is made to spin with an angular velocity $\sigma\Omega$ about the same diameter, the boundary conditions to be satisfied are

$$u = w = 0, \quad v = r\sin\theta\cdot\sigma\Omega, \quad \text{when } r = a,\ t > 0\,, \tag{8}$$

together with

$$u = w = 0, \quad v = r\sin\theta\cdot\Omega, \quad \text{when } r \to \infty\,. \tag{9}$$

3. - Simplification of the differential equations.

Let us assume

$$u = \Omega^2 r \sin\theta \cos\theta \cdot t\, f(\eta)\,, \tag{10}$$

$$v = \Omega\, r \sin\theta \cdot g(\eta)\,, \tag{11}$$

$$w = -2\,\Omega^2 \nu^{1/2} t^{3/2} (3 \cos^2 \theta - 1)\, h(\eta)\,, \tag{12}$$

where

$$\eta = \frac{r-a}{2\,(\nu\, t)^{1/2}}\,. \tag{13}$$

Substituting these values in (7), (5) and (6) we obtain

$$f'' + 2\,\eta\, f' - 4 = 4 - 4g^2 - 4\,\Omega^2 f^2 \cos 2\theta \cdot t^2 - \tag{14}$$

$$- 8\,\nu^{1/2} t^2 (3 \cos^2\theta - 1) h \left[\frac{\Omega^2 f \sqrt{t}}{r} - \frac{\Omega^2 f'\, t^{1/2}}{2\sqrt{\nu}}\right] - \frac{4\sqrt{\nu\, t}\, f'}{r}\,,$$

$$g'' + 2\,\eta\, g' = 4\, t^2 f\, g\, \Omega^2 \cos^2\theta - \tag{15}$$

$$- 8\,\Omega^2 \nu^{1/2} t^{5/2} (3 \cos^2\theta - 1)\, h \left[\frac{g}{r} + \frac{1}{2}\,\frac{g'}{(\nu\, t)^{1/2}}\right] + 4\,\Omega^2 \cos^2\theta \cdot t^2 f\, g - \frac{4\, g'}{r}\sqrt{\nu t}\,;$$

$$f - h' = 0\,. \tag{16}$$

During early stages of motion when t is small (or in boundary layer terminology: when the thickness of the boundary layer is small) we may neglect the terms containing powers of t. Therefore omitting such terms in the above equations we get to a first order of approximation the following equations:

$$f'' + 2\,\eta\, f' - 4\, f = 4 - 4\, g^2, \tag{17}$$

$$g'' + 2\,\eta\, g' = 0\,, \tag{18}$$

$$f = h', \tag{19}$$

where dashes denote differentiation with respect to η.

The boundary conditions now are

(20) $$f = 0 = h, \quad g = \sigma, \quad \text{when } \eta = 0\,,$$

(21) $$f = 0, \quad g = 1, \quad \text{when } \eta \to \infty\,.$$

These equations and the boundary conditions are the same as the equations in the problem of *Rotation of finite disc in rotating fluid, boundary layer growth: motion started impulsively from rest* by the present author.

4. - Solution.

Solution of equation (18) is

(22) $$g = A \operatorname{erf} \eta + B\,,$$

where $\operatorname{erf} \eta$ stands for

$$\frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2}\, d\eta\,.$$

Applying the boundary condition (20): $\sigma = B$ and $1 = A + B$, whence $A = 1 - \sigma$ and $B = \sigma$.

Thus equation (22) becomes

(23) $$g(\eta) = (1 - \sigma) \operatorname{erf} \eta + \sigma\,.$$

Substituting this value of $g(\eta)$ in (17) we get

(24) $$f'' + 2\eta f' - 4f = 4(1 - \sigma^2) - 8\sigma(1 - \sigma) \operatorname{erf} \eta - 4(1 - \sigma)^2 \operatorname{erf}^2 \eta\,.$$

Solution of (24) is

(25) $$f = A(1 + 2\eta^2) + B[(1 + 2\eta^2) \operatorname{erfc} \eta - (2/\sqrt{\pi})\, \eta\, e^{-\eta^2}] + 2(1 - \sigma^2)\eta^2 - $$
$$- 2\sigma(1 - \sigma)[\eta^2 \operatorname{erf} \eta - (1/2) \operatorname{erf} \eta + (\eta\, e^{-\eta^2}/\sqrt{\pi})] -$$
$$- 2(1 - \sigma)^2[(2/\sqrt{\pi})\eta\, e^{-\eta^2} \operatorname{erf} \eta + (1/\pi)e^{-2\eta^2} + \eta^2 \operatorname{erf}^2 \eta].$$

The constants A and B are to be determined by applying the boundary conditions (20) and (21). So we have

$$0 = A + B - 2(1-\sigma)^2 (1/\pi),$$

$$0 = A(1 + 2\eta^2) + 2(1-\sigma^2)\eta^2 - 2\sigma(1-\sigma)(\eta^2 - 1/2) - 2(1-\sigma)^2\eta^2 \quad \text{when } \eta \to \infty,$$

or

$$0 = \{A + \sigma(1-\sigma)\} + 2\eta^2 [A + 1 - \sigma^2 - \sigma(1-\sigma) - (1-\sigma)^2] \quad \text{when } \eta \to \infty,$$

which can be satisfied if

$$A = -\sigma(1-\sigma);$$

and therefore

$$B = \frac{(1-\sigma)[2 + \sigma(\pi - 2)]}{\pi}.$$

Substituting the values of A and B the equation (25) becomes

$$(26) \quad f = -\sigma(1-\sigma)(1+2\eta^2) + \frac{(1-\sigma)[2+\sigma(\pi-2)]}{\pi}\left[(1+2\eta^2)\operatorname{erfc}\eta - \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\right]$$
$$+ 2(1-\sigma^2)\eta^2 - 2\sigma(1-\sigma)[\eta^2 \operatorname{erf}\eta - (1/2)\operatorname{erf}\eta + (\eta e^{-\eta^2}/\sqrt{\pi})]$$
$$- 2(1-\sigma)^2[(2/\sqrt{\pi})\eta e^{-\eta^2}\operatorname{erf}\eta + (1/\pi)e^{-2\eta^2} + \eta^2 \operatorname{erf}^2\eta].$$

Since from equation (16) $f = h'$, hence function h is obtained by a quadrature of f.

Hence h is obtained as

$$h = -\sigma(1-\sigma)[\eta + (2/3)\eta^3]$$
$$+ \frac{(1-\sigma)[2+\sigma(\pi-2)]}{\pi}\left[\left[\eta + \frac{2}{3}\eta^3\right]\operatorname{erfc}\eta - \frac{2}{3\sqrt{\pi}}(1+\eta^2)e^{-\eta^2}\right] +$$

$$+\frac{2}{3}(1-\sigma^2)\,\eta^3-2\sigma(1-\sigma)\left[\left[\frac{1}{3}\,\eta^3-\frac{1}{2}\,\eta\right]\operatorname{erf}\eta+\frac{1}{3\sqrt{\pi}}\,\eta^2\,e^{-\eta^2}-\frac{2}{3\sqrt{\pi}}\,e^{-\eta^2}\right]$$

$$-2(1-\sigma)^2\left[\eta\left\{\frac{1}{\sqrt{\pi}}\,e^{-\eta^2}+\eta\operatorname{erf}\eta\right\}^2-\frac{2}{3\sqrt{\pi}}\operatorname{erf}\eta\cdot e^{-\eta^2}+\frac{2}{3\sqrt{\pi}}\operatorname{erf}(\sqrt{2}\eta)\right.$$

$$\left.-\frac{2}{3}\,\eta^3\operatorname{erf}^2\eta-\frac{4}{3\sqrt{\pi}}\,\eta^2\operatorname{erf}\eta\cdot e^{-\eta^2}-\frac{2}{3\pi}\,\eta\,e^{-2\eta^2}\right]+\text{constant}.$$

Since $h=0$ when $\eta=0$, we have

$$\text{constant}=\frac{2(1-\sigma)}{3\sqrt{\pi}}\,\frac{2-\sigma(\pi+2)}{\pi}.$$

Therefore the complete value of h is

$$h=-\sigma(1-\sigma)\left(\eta+(2/3)\eta^3\right)+ \tag{27}$$

$$+\frac{(1-\sigma)[2+\sigma(\pi-2)]}{\pi}\left[\left[\eta+\frac{2}{3}\,\eta^3\right]\operatorname{erfc}\eta-\frac{2}{3\sqrt{\pi}}\,(1+\eta^2)\,e^{-\eta^2}\right]$$

$$+\frac{2}{3}(1-\sigma^2)\eta^3-2\sigma(1-\sigma)\left[\left[\frac{1}{3}\,\eta^3-\frac{1}{2}\eta\right]\operatorname{erf}\eta+\frac{1}{3\sqrt{\pi}}\,\eta^2\,e^{-\eta^2}-\frac{2}{3\sqrt{\pi}}\,e^{-\eta^2}\right]$$

$$-2(1-\sigma)^2\left[\eta\left\{\frac{1}{\sqrt{\pi}}\,e^{-\eta^2}+\eta\operatorname{erf}\eta\right\}^2-\frac{1}{3\sqrt{\pi}}\operatorname{erf}\eta\cdot e^{-\eta^2}+\frac{2}{3\sqrt{\pi}}\operatorname{erf}(\sqrt{2}\eta)\right.$$

$$\left.-\frac{2}{3}\,\eta^3\operatorname{erf}^2\eta-\frac{4}{3\sqrt{\pi}}\,\eta^2\operatorname{erf}\eta\cdot e^{-\eta^2}-\frac{2}{3\pi}\,\eta\,e^{-2\eta^2}\right]+\frac{2(1-\sigma)}{3\sqrt{\pi}}\,\frac{2-\sigma(\pi+2)}{\pi}.$$

5. - Graphical representation.

The functions $f(\eta)$, $g(\eta)$, $h(\eta)$ have been discussed by the present author in detail for various values of σ. Appropriate graphs have also been drawn here (Fig. 1 and Fig. 2).

6. - Discussion of the results.

From the expressions of u, v, w given by (10), (11), (12) it is evident that u vanishes both on the axis of the rotation and in the equatorial plane, v vanis-

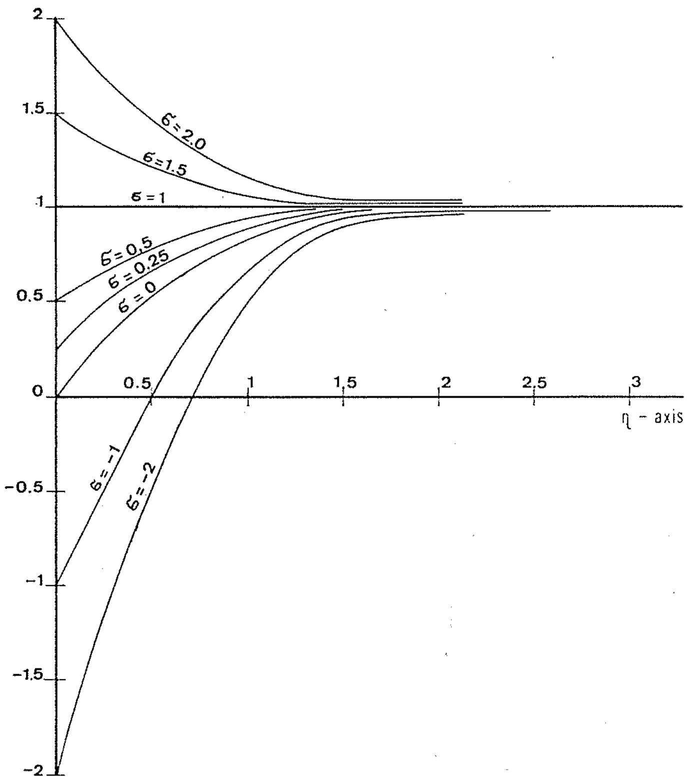

Fig. 1: Graphs of $g(\eta)$.

hes on the axis of rotation; the various values of σ, whether it is positive or negative, greater than 1 or less than 1, affect the inflow or outflow as discussed in the case of the rotating disc in the rotating liquid by the present author.

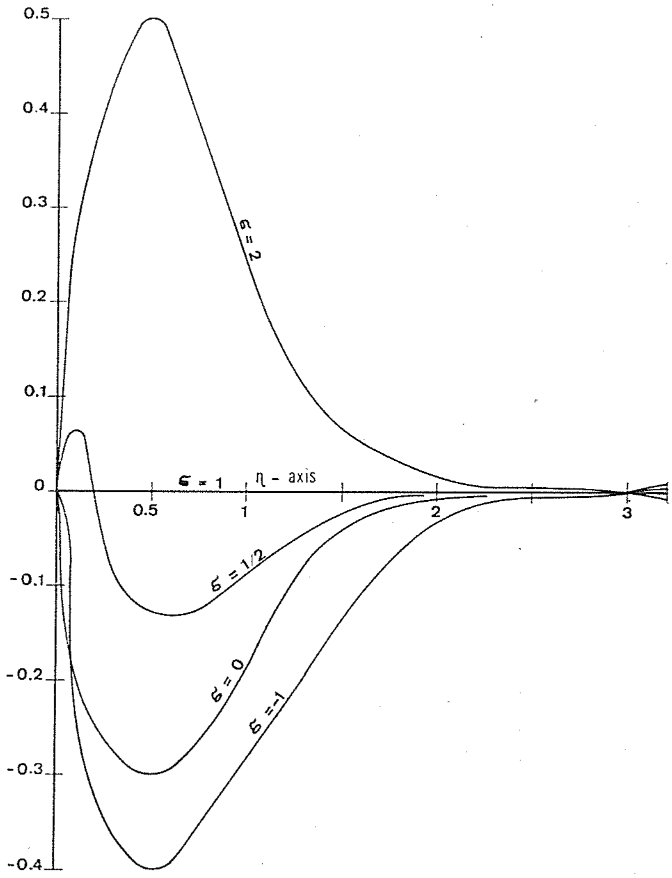

Fig. 2: Graphs of $f(\eta)$.

7. - Stream function.

A stream function may be defined by the equations

$$u = \frac{r}{\sin\theta}\frac{\partial\psi}{\partial r}, \qquad w = -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta},$$

whence the stream function may be expressed

$$\psi = 2\,\Omega^2\,t^{3/2}\,\nu^{1/2}\cos\theta\sin^2\theta\cdot h(\eta)\,.$$

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Summary.

The growth of motion of revolving flow in the earlier stages of its development caused by a finite sphere which at $t = 0$ is suddenly made to rotate with a constant angular spin in revolving liquids has been discussed. The various cases when the angular velocity of the disc is greater or less than the angular velocity of the rotating fluid have been discussed. The case when the sphere rotates in the opposite sense to that of the external flow has also been considered.

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