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A Correction and Remark on a Former Paper. (**)

In an earlier paper [1] it has been stated that

Theorem A. *If* $f_1(z) = \sum_0^\infty a_n z^n$, $f_2(z) = \sum_0^\infty b_n z^n$ *be integral functions of the same order* ϱ $(0 < \varrho < \infty)$ *and types* T_1 $(0 < T_1 < \infty)$ *and* T_2 $(0 < T_2 < \infty)$, *respectively, and* $f(z) = \sum_0^\infty c_n z^n$, *where* $|c_n| \sim |\sqrt{a_n b_n}|$, *then* $f(z)$ *is an integral function of order* ϱ *and type* T *such that*

$$T \leqslant \sqrt{T_1 T_2}\,.$$

That $f(z)$ must be of order ϱ has not been established and it has been pointed out by the Reviewer ([2], p. 430) that under the hypothesis of the above Theorem it is not possible to do so. Consequently, I restate the theorem as following:

Theorem B. *If* $f_1(z) = \sum_0^\infty a_n z^n$, $f_2(z) = \sum_0^\infty b_n z^n$ *be integral functions of the same order* ϱ $(0 < \varrho < \infty)$ *and type* T_1 $(0 < T_1 < \infty)$ *and* T_2 $(0 < T_2 < \infty)$, *respectively, and* $f(z) = \sum_0^\infty c_n z^n$, *where* $|c_n| \sim |\sqrt{a_n b_n}|$, *then* $f(z)$ *is an integral function of type* T *such that*

$$T \leqslant |\sqrt{T_1 T_2}|\,, \tag{1}$$

provided $f(z)$ *is of order* ϱ.

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The proof is omitted as it similar to that of Theorem 1 of my paper ([1], p. 251).

The condition that $f(z)$ should be order ϱ is also required for Corollary 2 ([1], p. 252) to be true.

We remark here that the condition that $f(z)$ should be of order ϱ is, however, not a necessary condition for the inequality in (1) to hold. In fact, we have

Theorem C. *There exist integral function $f_1(z)$, $f_2(z)$ and $f(z)$, as defined in Theorem A, such that the order of $f(z)$ is not equal to that of $f_1(z)$ and $f_2(z)$ and yet*

$$T < | \sqrt{T_1 T_2} | .$$

Proof. We take the example given by the Reviewer. Let

$$f(z) = \exp z^2 + \exp z = \sum_0^\infty a_n z^n, \qquad f_2(z) = z f_1(z) = \sum_0^\infty b_n z^n.$$

Then, for $m = 0, 1, 2, \ldots$, we have

$$a_{2m} = 1/m! + 1/(2m)!, \qquad a_{2m+1} = 1/(2m+1)!,$$

$$b_0 = 0, \qquad b_{2m} = 1/(2m-1)! \quad (m \neq 0), \qquad b_{2m+1} = 1/m! + 1/(2m)!.$$

Here the orders of $f_1(z)$ and $f_2(z)$ are both $\varrho = 2$ and their types $T_1 = T_2 = 1$.

Now, let

$$f(z) = \sum_0^\infty c_n z^n = \sum_0^\infty (a_n b_n)^{1/2} z^n .$$

Then, using the formulae for the order and type ([3]: pp. 9, 11) it can be easily shown that the order of $f(z)$ is

$$\varrho' = \limsup_{n \to \infty} \frac{n \log n}{\log | c_n |^{-1}} = \frac{4}{3} \neq \varrho$$

and its type is

$$T = \limsup_{n \to \infty} \left[\frac{n}{e \varrho'} | c_n |^{\varrho'/n} \right] = \frac{3}{4} 2^{1/3} < 1 .$$

Hence, $T < | \sqrt{T_1 T_2} |$.

References.

[1] R. S. L. Srivastava, *On the order and type of integral functions*, Riv. Mat. Univ. Parma **10** (1959), 249 - 255.

[2] S. M. Shah, *Review* No. 2204, Math. Reviews **25** (1963), p. 430.

[3] R. P. Boas jr., *Entire functions*, Acad. Press, Inc. 1954.

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