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Summability Factors for Generalized Strong Riesz Logarithmic Boundedness. (**)

1.1. - Definitions and notations.

Let $\sum a_n$ be a given infinite series, and let $s_n^\varkappa$ denote the nth Cesàro mean of order $\varkappa$ ($\varkappa > -1$) of the sequence $\{s_n\}$, where $s_n = s_n^0$ is the nth partial sum of the series $\sum a_n$.

If

$$\sum_{\nu=1}^{n} \frac{|s_\nu|}{\nu} = O(\log n)\,, \tag{1.1.1}$$

as $n \to \infty$, then $\sum a_n$ is said to be strongly bounded by logarithmic means with index 1, or bounded [R, $\log n$, 1], symbolically

$$\sum a_n = O(1)\,[\mathrm{R},\ \log n,\ 1]\,.$$

If for $\varkappa > -1$,

$$\sum_{\nu=1}^{n} \frac{|s_\nu^\varkappa|}{\nu} = O(\log n)\,, \tag{1.1.2}$$

as $n \to \infty$, we shall write, by analogy,

$$\sum a_n = O(1)\,[(\mathrm{R},\ \log n,\ 1)\,(\mathrm{C},\ \varkappa)].$$

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We observe that when $\varkappa = 0$, summability (C, 0) is the same as convergence, and (1.1.2), in that case, reduces to (1.1.1). Thus boundedness [(R, log n, 1) (C, 0)] is the same as boundedness [R, log n, 1] .

For any sequence $\{\lambda_n\}$, we write throughout, for $n = 1, 2, 3, \ldots$,

$$\Delta^0 \lambda_n = \lambda_n, \qquad \Delta\lambda_n = \Delta^1 \lambda_n = \lambda_n - \lambda_{n+1}, \qquad \Delta^2 \lambda_n = \Delta(\Delta\lambda_n)\,,$$

and in general,

$$\Delta^p \lambda_n = \Delta(\Delta^{p-1} \lambda_n), \qquad \text{for} \qquad p \geqslant 1\,.$$

We have the useful identity (σ integral)

$$\Delta^\sigma \varepsilon_n = \sum_{j=0}^{\sigma} \binom{\sigma}{j} (-1)^j \ \varepsilon_{n+j}\,. \tag{1.1.3}$$

1.2. - Introduction.

The well known consistency theorem for CESÀRO summability asserts that every infinite series which is summable (C, $\varkappa$), $\varkappa > -1$, is also summable (C, $\varkappa'$) for $\varkappa' > \varkappa$, symbolically,

$$(\mathrm{C},\ \varkappa) \subset (\mathrm{C},\ \varkappa')\,, \qquad \varkappa' > \varkappa > -1\,.$$

It is natural to expect such a consistency theorem also for the *generalized strong Riesz logarithmic boundedness*, defined above as boundedness [(R, log n, 1) (C, $\varkappa$)]; in other words, it is natural to expect the result that if $\sum a_n = O(1)$ [(R, log n, 1) (C, $\varkappa$)], $\varkappa > -1$, then $\sum a_n = O(1)$ [(R, log n, 1) (C, $\varkappa'$)] for every $\varkappa' > \varkappa$.

In this paper, in the form of Theorem 1, we establish this result for the case $\varkappa > -1$.

As can be easily verified by considering the series $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot n$, boundedness [(R, log n, 1) (C, 1)] may be true and yet boundedness [R, log n, 1] may not be true. Hence arises the question of choosing such factors ε_n as will make the series $\sum a_n \varepsilon_n$ bounded [(R, log n, 1) (C, $\varkappa$)] whenever $\sum a_n$ is bounded [(R, log n, 1) (C, $\varkappa'$)] for $\varkappa < \varkappa'$.

We prove, in the form of Theorem 2, a result which provides an answer to this question in the case in which $\varkappa$ and $\varkappa'$ are both positive integers and $0 \leqslant \varkappa < \varkappa'$.

2.1. - We establish the following theorems.

Theorem 1. *If* $\varkappa > -1$, *and* $\sum a_n = O(1)$ $[(\mathrm{R}, \log n, 1)\ (\mathrm{C}, \varkappa)]$, *then* $\sum a_n = O(1)$ $[(\mathrm{R}, \log n, 1)\ (\mathrm{C}, \varkappa')]$, *for every* $\varkappa' > \varkappa$.

Theorem 2. *If* $\varkappa$ *and* $\varkappa'$ *are positive integers such that*: $0 \leqslant \varkappa < \varkappa'$, *and* $\sum a_n = O(1)$ $[\mathrm{R}, \log n, 1)\ (\mathrm{C}, \varkappa')]$, *and if the factor sequence* $\{\varepsilon_n\}$ *satisfies the conditions*:

(2.1.1) $$|\varepsilon_n| \equiv \bar{\varepsilon}_n \text{ is monotonic non-increasing,}$$

(2.1.2) $$\sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{\varkappa'-\varkappa} \cdot |\varepsilon_\mu| = O(\log n),$$

as $n \to \infty$, *and*

(2.1.3) $$\sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{\varkappa'+1} \cdot |\Delta^{\varkappa'+2} \varepsilon_\mu| = O(1),$$

as $n \to \infty$, *then* $\sum a_n \varepsilon_n = O(1)$ $[(\mathrm{R}, \log n, 1)\ (\mathrm{C}, \varkappa)]$.

2.2. – We need the following lemmas.

Lemma 1 (1). *If* $\sigma > -1$, $\sigma - \delta > 0$, *then*

$$\sum_{n=\mu}^{\infty} \frac{A_{n-\mu}^{\delta}}{n\, A_n^{\sigma}} = \sum_{n=0}^{\infty} \frac{A_n^{\delta}}{(n+\mu)\, A_{n+\mu}^{\sigma}} = \frac{1}{\mu \cdot A_\mu^{\sigma-\delta-1}}.$$

Lemma 2. *If* $\delta > 0$, *and* $\{\bar{\varepsilon}_\mu\} \equiv \{|\varepsilon_\mu|\}$ *is a monotonic non-increasing sequence such that*

$$\sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{\delta} \cdot |\varepsilon_\mu| = O(\log n),$$

as $n \to \infty$, *then* $\varepsilon_\mu = o(1)$.

Proof. Let $\varepsilon_\mu \neq o(1)$. Then $\bar{\varepsilon}_\mu \neq o(1)$, that is, $\lim_{\mu\to\infty} \bar{\varepsilon}_\mu \neq 0$. But this limit which surely exists, since $\{\bar{\varepsilon}_\mu\}$ is monotonic non-increasing and with rough lower bound zero, is greater than or equal to zero. But as stated above

(1) Chow [1], Lemma 1. Also cf. Peyerimhoff [3], p. 418, footnote (3).

it is not zero. Hence $\lim_{\mu\to\infty} \bar{\varepsilon}_\mu$ is greater than zero. Call it δ_1. Then each $\bar{\varepsilon}_\mu \geqslant \delta_1 > 0$, and therefore,

$$\sum_{\mu=1}^{n} \log \mu \cdot A_\mu^\delta \cdot \mid \varepsilon_\mu \mid \geqslant \delta_1 \cdot \sum_{\mu=1}^{n} \log \mu \cdot A_\mu^\delta$$

$$> K \cdot \sum_{\mu=1}^{n} \log \mu > K \cdot n \cdot \log n \quad (^2) .$$

This completes the proof of the lemma.

Lemma 3. *If r is a positive integer, $\Delta^{r+1} \varepsilon_\nu = o(1)$, and*

$$\sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{r+1} \cdot \mid \Delta^{r+2} \varepsilon_\mu \mid = O(1),$$

as $n \to \infty$, then

$$\sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{p+1} \cdot \mid \Delta^{p+2} \varepsilon_\mu \mid = O(1) ,$$

as $n \to \infty$, for every integer p such that $-1 < p < r$.

Proof. Since $\Delta^{r+1} \varepsilon_\nu = o(1)$, we can write

$$\sum_{\nu=\mu}^{\infty} \Delta^{r+2} \varepsilon_\nu = \sum_{\nu=\mu}^{\infty} (\Delta^{r+1} \varepsilon_\nu - \Delta^{r+1} \varepsilon_{\nu+1}) = \Delta^{r+1} \varepsilon_\mu .$$

Therefore, we have

$$\sum_{\mu=1}^{n} \log \mu \cdot A_\mu^r \cdot \mid \Delta^{r+1} \varepsilon_\mu \mid \leqslant \sum_{\mu=1}^{n} \log \mu \cdot A_\mu^r \cdot \sum_{\nu=\mu}^{\infty} \mid \Delta^{r+2} \varepsilon_\nu \mid$$

$$< \sum_{\nu=1}^{\infty} \mid \Delta^{r+2} \varepsilon_\nu \mid \cdot \sum_{\mu=1}^{\nu} \log \mu \cdot A_\mu^r < \sum_{\nu=1}^{\infty} \mid \Delta^{r+2} \varepsilon_\nu \mid \cdot \log \nu \cdot \sum_{\mu=1}^{\nu} A_\mu^r$$

$$< \sum_{\nu=1}^{\infty} \mid \Delta^{r+2} \varepsilon_\nu \mid \cdot \log \nu \cdot A_\nu^{r+1} = O(1) ,$$

as $n \to \infty$, by the hypothesis.

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(2) Throughout this paper, K denotes an absolute positive constant, not necessarily the same at each occurrence. Cf. PATI [**2**], p. 295, line 15, first inequality.

Similarly, by repeated application of the above process, we can have

$$\sum_{\mu=1}^{n} \log \mu \cdot A_{\mu}^{p+1} \cdot \left| \Delta^{p+2} \varepsilon_{\mu} \right| = O(1) ,$$

as $n \to \infty$, for $-1 < p < r$.

This completes the proof of the Lemma 3.

Lemma 4. *If* $\delta > 0$ *and*

$$\sum_{\mu=1}^{n} \log \mu \cdot A_{\mu}^{\delta} \cdot \left| \varepsilon_{\mu} \right| = O(\log n) ,$$

as $n \to \infty$, *then*

$$\sum_{\mu=1}^{n} \log \mu \cdot A_{\mu}^{\delta} \cdot \left| \Delta^{\sigma} \varepsilon_{\mu} \right| = O(\log n) ,$$

as $n \to \infty$, *for every integer* $\sigma \geqslant 0$.

Proof. Since $\Delta^{\sigma} \varepsilon_n = \sum_{j=0}^{\sigma} \binom{\sigma}{j} (-1)^j \cdot \varepsilon_{n+j}$, we have

$$\sum_{\mu=1}^{n} \log \mu \cdot A_{\mu}^{\delta} \cdot \left| \Delta^{\sigma} \varepsilon_{\mu} \right| \leqslant \sum_{\mu=1}^{n} \log \mu \cdot A_{\mu}^{\delta} \cdot \sum_{j=0}^{\sigma} \binom{\sigma}{j} \left| \varepsilon_{\mu+j} \right|$$

$$= \sum_{j=0}^{\sigma} \binom{\sigma}{j} \sum_{\mu=1}^{n} \log \mu \cdot A_{\mu}^{\delta} \cdot \left| \varepsilon_{\mu+j} \right| = O(\log n) ,$$

as $n \to \infty$, by hypothesis.

3.1. – Proof of Theorem 1. We have to show that, if (1.1.2) holds for every $\varkappa > -1$, then

$$\sum_{\nu=1}^{n} \frac{\left| s_{\nu}^{\varkappa'} \right|}{\nu} = O(\log n) , \tag{3.1.1}$$

as $n \to \infty$.

We have

$$\sum_{\nu=1}^{n} \frac{\left| s_{\nu}^{\varkappa'} \right|}{\nu} = \sum_{\nu=1}^{n} (\nu A_{\nu}^{\varkappa'})^{-1} \left| S_{\nu}^{\varkappa'} \right| = \sum_{\nu=1}^{n} (\nu A_{\nu}^{\varkappa'})^{-1} \left| \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{\varkappa'-\varkappa-1} \cdot (A_{\mu}^{\varkappa} \cdot s_{\mu}^{\varkappa}) \right|$$

$$\leqslant \sum_{\nu=1}^{n} (\nu A_{\nu}^{\varkappa'})^{-1} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{\varkappa'-\varkappa-1} \cdot A_{\mu}^{\varkappa} \cdot \left| s_{\mu}^{\varkappa} \right|$$

$$= \sum_{\mu=1}^{n} A_{\mu}^{\varkappa} \cdot \left| s_{\mu}^{\varkappa} \right| \sum_{\nu=\mu}^{n} \frac{A_{\nu-\mu}^{\varkappa'-\varkappa-1}}{\nu A_{\nu}^{\varkappa'}}$$

$$\leqslant K \cdot \sum_{\mu=1}^{n} A_\mu^\varkappa \cdot \left| s_\mu^\varkappa \right| (\mu \cdot A_\mu^\varkappa)^{-1}, \qquad \text{by Lemma 1,}$$

$$= O(\log n),$$

as $n \to \infty$, by hypothesis.

This completes the proof of Theorem 1.

3.2. – Proof of Theorem 2. We have to show that, if (3.1.1) holds for an integer $\varkappa'$ $(\varkappa' > \varkappa \geqslant 0)$, then

$$\sum_{\nu=1}^{n} \frac{|\tilde{s}_\nu^\varkappa|}{\nu} = O(\log n), \tag{3.2.1}$$

as $n \to \infty$, where $\tilde{s}_\nu^\varkappa$ is the νth CESÀRO mean of an integer order $\varkappa$ of the factored series $\sum a_n \varepsilon_n$.

Writing $\tilde{S}_\nu^\varkappa$ for the νth CESÀRO sum of order $\varkappa$ of the series $\sum a_n \varepsilon_n$, we have

$$\tilde{S}_\nu^\varkappa = \sum_{\mu=0}^{\nu} A_{\nu-\mu}^\varkappa \cdot (a_\mu \, \varepsilon_\mu)$$

$$= \sum_{\mu=0}^{\nu-1} \Delta_\mu \, (A_{\nu-\mu}^\varkappa \cdot \varepsilon_\mu) \sum_{\lambda=0}^{\mu} a_\lambda + \left[A_{\nu-\mu}^\varkappa \cdot \varepsilon_\mu \cdot \sum_{\lambda=0}^{\mu} a_\lambda \right]_{\mu=\nu}$$

$$= \sum_{\mu=0}^{\nu} \Delta_\mu^1 \, (A_{\nu-\mu}^\varkappa \cdot \varepsilon_\mu) \, S_\mu, \qquad \text{by partial summation once,}$$

. . . .

$$= \sum_{\mu=0}^{\nu} S_\mu^{\varkappa'} \cdot \Delta_\mu^{\varkappa'+1} (A_{\nu-\mu}^\varkappa \cdot \varepsilon_\mu), \qquad \text{by repeated partial summation } \varkappa'+1 \text{ times, where } \varkappa' = 1, 2, 3, \ldots .$$

Therefore,

$$\sum_{\nu=1}^{n} \frac{|\tilde{s}_\nu^\varkappa|}{\nu} = \sum_{\nu=1}^{n} (\nu \, A_\nu^\varkappa)^{-1} \left| \sum_{\mu=1}^{\nu} S_\mu^{\varkappa'} \, \Delta_\mu^{\varkappa'+1} (A_{\nu-\mu}^\varkappa \cdot \varepsilon_\mu) \right| + K = \sum + K.$$

Now

$$\sum \leqslant \sum_{\nu=1}^{n} (\nu \, A_\nu^\varkappa)^{-1} \sum_{\mu=1}^{\nu} \frac{|s_\mu^{\varkappa'}|}{\mu} \cdot \mu \cdot A_\mu^{\varkappa'} \left| \Delta_\mu^{\varkappa'+1} (A_{\nu-\mu}^\varkappa \cdot \varepsilon_\mu) \right|$$

$$= \sum_{\nu=1}^{n} (\nu \, A_\nu^\varkappa)^{-1} \left[\sum_{\mu=0}^{\nu-1} \Delta_\mu \left(\mu \cdot A_\mu^{\varkappa'} \left| \Delta_\mu^{\varkappa'+1} (A_{\nu-\mu}^\varkappa \cdot \varepsilon_\mu) \right| \right) \sum_\mu + \left(\mu \cdot A_\mu^{\varkappa'} \left| \Delta_\mu^{\varkappa'+1} (A_{\nu-\mu}^\varkappa \cdot \varepsilon_\mu) \right| \sum_\mu \right)_{\mu=\nu} \right]$$

(by partial summation, where $\sum_\mu = \sum_{\lambda=1}^{\mu} \lambda^{-1} \cdot \left| s_\lambda^{\varkappa'} \right| = O(\log \mu)$),

$$\leqslant K \sum_{\nu=1}^{n} (\nu\, A_\nu^\varkappa)^{-1} \cdot \sum_{\mu=1}^{\nu} \Delta_\mu \left(\mu \cdot A_\mu^{\varkappa'} \left| \Delta_\mu^{\varkappa'+1} (A_{\nu-\mu}^\varkappa \cdot \varepsilon_\mu) \right| \right) \cdot \log \mu \text{ (3)}$$

$$\leqslant K \sum_{\nu=1}^{n} (\nu\, A_\nu^\varkappa)^{-1} \cdot \sum_{\mu=1}^{\nu} \mu \cdot \log \mu \cdot A_\mu^{\varkappa'} \left| \Delta_\mu^{\varkappa'+2} (A_{\nu-\mu}^\varkappa \cdot \varepsilon_\mu) \right|$$

$$+ K \sum_{\nu=1}^{n} (\nu\, A_\nu^\varkappa)^{-1} \cdot \sum_{\mu=1}^{\nu} \mu \cdot \log \mu \cdot A_\mu^{\varkappa'-1} \left| \Delta^{\varkappa'+1} (A_{\nu-(\mu+1)}^\varkappa \cdot \varepsilon_{\mu+1}) \right|$$

$$+ K \sum_{\nu=1}^{n} (\nu\, A_\nu^\varkappa)^{-1} \cdot \sum_{\mu=1}^{\nu} \log \mu \cdot A_{\mu+1}^{\varkappa'} \left| \Delta^{\varkappa'+1} (A_{\nu-(\mu+1)}^\varkappa \cdot \varepsilon_{\mu+1}) \right|$$

$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3, \quad \text{say.} \tag{3.2.2}$$

Let us start with

$$\sum\nolimits_1 = \sum_{\nu=1}^{n} (\nu\, A_\varkappa^\nu)^{-1} \cdot \sum_{\mu=1}^{\nu} \mu \cdot \log \mu \cdot A_\mu^{\varkappa'} \left| \sum_{r=0}^{\varkappa'+2} \binom{\varkappa'+2}{r} \Delta_\mu^r (A_{\nu-\mu}^\varkappa) \cdot \Delta^{\varkappa'+2-r} \varepsilon_{\mu+r} \right|$$

$$\leqslant \sum_{r=0}^{\varkappa'+2} \binom{\varkappa'+2}{r} \sum_{\nu=1}^{n} (\nu\, A_\nu^\varkappa)^{-1} \cdot \sum_{\mu=1}^{\nu} \log \mu \cdot A_\mu^{\varkappa'+1} \left| A_{\nu-\mu}^{\varkappa-r} \right| \left| \Delta^{\varkappa'+2-r} \varepsilon_{\mu+r} \right|, \tag{3.2.3}$$

where we consider, for the range of values of r, three cases as follows.

Case 1: $r = 0$. In this case $\sum_1$ reduces to

$$\sum_{\nu=1}^{n} (\nu\, A_\nu^\varkappa)^{-1} \cdot \sum_{\mu=1}^{\nu} \log \mu \cdot A_\mu^{\varkappa'+1} \cdot A_{\nu-\mu}^\varkappa \left| \Delta^{\varkappa'+2} \varepsilon_\mu \right| = O(\log n),$$

as $n \to \infty$, if

$$\sum_{\mu=1}^{\nu} \log \mu \cdot A_\mu^{\varkappa'+1} \cdot A_{\nu-\mu}^\varkappa \cdot \left| \Delta^{\varkappa'+2} \varepsilon_\mu \right| = O(A_\nu^\varkappa),$$

that is, if

$$\sum_{\mu=1}^{\nu} A_{\nu-\mu}^{\varkappa-1} \cdot \sum_{\lambda=1}^{\mu} \log \lambda \cdot A_\lambda^{\varkappa'+1} \left| \Delta^{\varkappa'+2} \varepsilon_\lambda \right| = O(A_\nu^\varkappa) \quad \text{(by partial summation)},$$

which holds since

$$\sum_{\lambda=1}^{\mu} \log \lambda \cdot A_\lambda^{\varkappa'+1} \cdot \left| \Delta^{\varkappa'+2} \varepsilon_\lambda \right| = O(1), \tag{3.2.4}$$

as $\mu \to \infty$, by virtue of the condition (2.1.3) of the hypothesis.

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(3) Throughout $\log \mu$ at $\mu = 1$ should be understood to be a positive constant, and not the routine zero.

Case 2: $r = 1, 2, \ldots, \varkappa + 1$. In this case a typical term in $\sum_1$ is

$$\binom{\varkappa' + 2}{r} \sum_{\nu=1}^{n} (\nu\, A_\nu^\varkappa)^{-1} \sum_{\mu=1}^{\nu} \log \mu \cdot A_\mu^{\varkappa'+1} \cdot A_{\nu-\mu}^{\varkappa-r} \left| \Delta^{\varkappa'+2-r}\, \varepsilon_{\mu+r} \right|$$

$$\leqslant K \cdot \sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{\varkappa'+1} \left| \Delta^{\varkappa'+2-r}\, \varepsilon_{\mu+r} \right| \sum_{\nu=\mu}^{n} \frac{A_{\nu-\mu}^{\varkappa-r}}{\nu\, A_\nu^\varkappa}$$

$$\leqslant K \cdot \sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{\varkappa'+1} \cdot \left| \Delta^{\varkappa'+2-r}\, \varepsilon_{\mu+r} \right| (\mu \cdot A_\mu^{r-1})^{-1}$$

(by Lemma 1, since $0 < r \leqslant \varkappa + 1$),

$$= K \cdot \sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{\varkappa'+1-r} \left| \Delta^{\varkappa'+2-r}\, \varepsilon_{\mu+r} \right|$$

(3.2.5) $$= O(\log n)\,,$$

as $n \to \infty$, by Lemma 3, in virtue of the conditions (2.1.1), (2.1.2), (2.1.3) and Lemma 2.

Case 3: $r = \varkappa + 2, \varkappa + 3, \ldots, \varkappa' + 2$. In this case a typical term in $\sum_1$ is

$$\binom{\varkappa' + 2}{r} \sum_{\nu=1}^{n} (\nu\, A_\nu^\varkappa)^{-1} \sum_{\mu=1}^{\nu} \log \mu \cdot A_\mu^{\varkappa'+1} \cdot \left| A_{\nu-\mu}^{\varkappa-r} \right| \left| \Delta^{\varkappa'+2-r}\, \varepsilon_{\mu+r} \right|$$

$$\leqslant K \cdot \sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{\varkappa'+1} \left| \Delta^{\varkappa'+2-r}\, \varepsilon_{\mu+r} \right| \cdot \sum_{\nu=\mu}^{n} \frac{A_{\nu-\mu}^{-(r-\varkappa)}}{\nu\, A_\nu^\varkappa}$$

$$< K \cdot \sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{\varkappa'+1} \left| \Delta^{\varkappa'+2-r}\, \varepsilon_{\mu+r} \right| (\mu\, A_\mu^\varkappa)^{-1} \sum_{\nu=0}^{n-\mu} \left| A_\nu^{-(r-\varkappa)} \right|$$

$$= K \cdot \sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{\varkappa'-\varkappa} \left| \Delta^{\varkappa'+2-r}\, \varepsilon_{\mu+r} \right| \cdot \sum_{\nu=0}^{r-\varkappa-1} \left| A_\nu^{-(r-\varkappa)} \right|$$

(since $A_n^{-\alpha} = 0$ for $n \geqslant \alpha = 1, 2, \ldots$)

$$\leqslant K \cdot \sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{\varkappa'-\varkappa} \cdot \left| \Delta^{\varkappa'+2-r}\, \varepsilon_{\mu+r} \right|$$

(3.2.6) $$= O(\log n)\,,$$

as $n \to \infty$, by Lemma 4, in virtue of the condition (2.1.2) of the hypotheses.

Next, we consider $\sum_2$, where

$$\sum\nolimits_2 = \sum_{\nu=1}^{n} (\nu\, A_\nu^\varkappa)^{-1} \sum_{\mu=1}^{\nu} \mu \cdot \log \mu \cdot A_\mu^{\varkappa'-1} \left| \Delta^{\varkappa'+1} (A_{\nu-(\mu+1)}^{\varkappa} \cdot \varepsilon_{\mu+1}) \right|$$

$$\leqslant K \cdot \sum_{\nu=1}^{n} (\nu A_\nu^{\varkappa})^{-1} \cdot \sum_{\mu=1}^{\nu-1} \log (\mu + 1) \cdot A_{\mu+1}^{\varkappa'} \cdot \left| \Delta^{\varkappa'+1} (A_{\nu-(\mu+1)} \, \varepsilon_{\mu+1}) \right|$$

$$(\text{since } A_{\nu-(\mu+1)}^{\varkappa} = 0 \quad \text{when } \mu = \nu),$$

$$= K \cdot \sum_{\nu=1}^{n} (\nu A_\nu^{\varkappa})^{-1} \cdot \sum_{\mu=1}^{\nu} \log \mu \cdot A_\mu^{\varkappa'} \left| \Delta^{\varkappa'+1} (A_{\nu-\mu}^{\varkappa} \cdot \varepsilon_\mu) \right|$$

$$\leqslant K \cdot \sum_{r=0}^{\varkappa'+1} \binom{\varkappa'+1}{r} \cdot \sum_{\nu=1}^{n} (\nu A_\nu^{\varkappa})^{-1} \cdot \sum_{\mu=1}^{\nu} \log \mu \cdot A_\mu^{\varkappa'} \left| A_{\nu-\mu}^{\varkappa-r} \right| \left| \Delta^{\varkappa'+1-r} \, \varepsilon_{\mu+r} \right|. \tag{3.2.7}$$

We observe that (3.2.7) is the same as (3.2.3) with $\varkappa'$ replaced by $\varkappa' - 1$. So again, we have the following three cases as above.

Case 1: $r = 0$. We proceed exactly in the same manner as in Case 1 of $\sum_1$ and, finally, have only to show that

$$\sum_{\lambda=1}^{\mu} \log \lambda \cdot A_\lambda^{\varkappa'} \left| \Delta^{\varkappa'+1} \, \varepsilon_\lambda \right| = O(1), \tag{3.2.8}$$

as $\mu \to \infty$, which is true by Lemma 3, in virtue of the conditions (2.1.1), (2.1.2), (2.1.3) and Lemma 2.

Case 2: $r = 1, 2, \dots, \varkappa + 1$. For these values of r, starting with a typical term in (3.2.7) and working out as in Case 2 of $\sum_1$, finally we have only to show that

$$\sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{\varkappa'-r} \left| \Delta^{\varkappa'+1-r} \, \varepsilon_{\mu+r} \right| = O(\log n), \tag{3.2.9}$$

as $n \to \infty$, which is, *a fortiori*, true since (3.2.5) is true.

Case 3: $r = \varkappa + 2, \varkappa + 3, \dots, \varkappa' + 1$. In this case also, proceeding as in Case 3 of $\sum_1$, finally we have only to show that

$$\sum_{\mu=1}^{n} \log \mu \cdot A_\mu^{\varkappa'-\varkappa-1} \left| \Delta^{\varkappa'+1-r} \, \varepsilon_{\mu+r} \right| = O(\log n), \tag{3.2.10}$$

as $n \to \infty$, which is also, *a fortiori*, true since (3.2.6) is true.

Finally, a similar order-estimate for $\sum_3$ holds as for $\sum_2$, for,

$$\sum\nolimits_3 = \sum_{\nu=1}^{n} (\nu A_\nu^{\varkappa})^{-1} \cdot \sum_{\mu=1}^{\nu} \log \mu \cdot A_{\mu+1}^{\varkappa'} \cdot \left| \Delta^{\varkappa'+1} (A_{\nu-(\mu+1)}^{\varkappa} \cdot \varepsilon_{\mu+1}) \right|$$

$$\leqslant K \cdot \sum_{\nu=1}^{n} (\nu A_\nu^\varkappa)^{-1} \sum_{\mu=1}^{\nu-1} \log(\mu+1) \cdot A_{\mu+1}^{\varkappa'} \left| \Delta^{\varkappa'+1} (A_{\nu-(\mu+1)}^{\varkappa} \cdot \varepsilon_{\mu+1}) \right|$$

$$= K \cdot \sum_{\nu=1}^{n} (\nu A_\nu^\varkappa)^{-1} \sum_{\mu=1}^{\nu} \log \mu \cdot A_\mu^{\varkappa'} \left| \Delta^{\varkappa'+1} (A_{\nu-\mu}^{\varkappa} \cdot \varepsilon_\mu) \right| = K \cdot \sum\nolimits_2 .$$

This completes the proof of Theorem 2.

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