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The Classical Ergodic Problem and the Restricted One: a General Solution. (**)

1. - Introduction.

The aim of the present paper is to clarify the meaning of a recent result obtained by B. FORTE (see [6]) from the point of view of the statistical mechanics. We shall remember first some definitions and properties of dynamical systems.

Consider a dynamical olonomic system with n degrees of freedom. The system is described by n generalized coordinates $(q_1, \ldots, q_n)$ and the generalized momenta $(p_1, \ldots, p_n)$. Let Ω be the set of points $\omega \equiv (q_1, \ldots, q_n, p_1, \ldots, p_n)$ of the space R^{2n} consisting of real numbers and defining possible states for the system. The motion of the system is described by the classical HAMILTON's equations. Let be H the Hamiltonian of the system; we shall suppose that:

a) H doesn't depend on t explicitly;

b) H is such that for every point $\omega \in \Omega$ the equations of motion have a unique solution.

As a consequence, for each point $\omega \in \Omega$ and for every value of the variable t there is a unique point $\omega_t = T_t\,\omega$ which characterizes the state of the system at the time t with respect to the initial state defined in Ω by the point ω. Further the transformations T_t are an abelian group of one parameter transformations (see [1], p. 56).

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We shall associate to Ω the structure of a measure space by introducing a σ-algebra $\mathscr{F}$ of subsets of Ω (events), (e.g. the σ-algebra of L-measurable subsets, if such is Ω) and define on this σ-algebra a measure of probability m. We shall suppose that this measure of probability is invariant under the group of transformations T_t.

A random dynamical system in statistical equilibrium is a system defined by the space Ω (phase space), by the family of transformations T_t which characterizes the motion of the representative points in the phase space, by the set of events $\mathscr{F}$ and by the invariant measure of probability m.

Consider now any microscopic state function of the system, or an arbitrary measurable mapping of $(\Omega, \mathscr{F})$ into $(R_1, \mathscr{B})$, where R_1 is the set of real numbers, $\mathscr{B}$ the σ-algebra of the BOREL sets of R_1; we shall denote it by $f(\omega)$. Thus the concept of microscopic state function is the same as the concept of random variable on the probability space $(\Omega, \mathscr{F}, m)$.

By time average of the function $f(\omega)$ along the path passing through the point $\omega \in \Omega$ we mean the limit, if such exists:

$$\lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau f(T_t\, \omega)\, \mathrm{d}t;$$

let denote it by $\hat{f}(\omega)$.

What is said about the group of transformations T_t can be restricted to the abelian subgroup of the powers $\{T^n\}$ of a transformation T which coincides with a transformation T_t at a fixed value of the parameter t (see [3], p. 3).

Remember that, if V is an invariant part of finite measure of Ω, i.e. such that $T^{-1}\, V = V$ and $m\, V < +\infty$, and $f(\omega)$ is integrable on V and defined for each point $P \in V$, then $\hat{f}(\omega)$ exists almost everywhere on V and depends only on the path.

If V is an invariant part of finite measure of Ω, we define phase average of the function $f(\omega)$ on V the quantity, if such exists:

$$\bar{f} = \frac{1}{m(V)} \int_V f(\omega)\, \mathrm{d}V\,.$$

We shall say with KHINCHIN (see [2], p. 66) that the function $f(\omega)$ is ergodic on V if for almost all paths in V the time average is the same, and coincides with the phase average, i.e. if $\hat{f} = \bar{f}$.

The ergodic problem consists in finding ergodic phase functions associated to a given dynamical random system, i.e. phase functions which generate, by their average, the macroscopic state functions (see [1], p. 10).

The classical ergodic problem is the following: characterize the dynamical systems for which all integrable microscopic state functions are ergodic. The solution is given by the BIRKOFF's theorem:

Let V be an invariant part of finite measure of Ω. Every integrable microscopic function is ergodic if and only if V is metrically indecomposable, i.e. V can't be represented in the form $V = V' + V''$, where V' and V'' are invariant disjointed parts of positive measure.

The BIRKOFF's theorem doesn't characterize, for a given dynamical system, the ergodic microscopic state functions; it characterizes the dynamical random systems for which every microscopic state function is ergodic.

The restricted ergodic problem, on the contrary, consists in finding all the microscopic state functions which are ergodic for a given dynamical random system. About such problem, a first successful result has been obtained only by sufficient conditions (see [2], [5], [7]).

The point of views of the classical ergodic problem and the restricted one are quite different. We believe, following a recent result (see [6]), that these problems may be solved together. In fact, our design is to elaborate the contents of the paper by B. FORTE (see [6]) and to show that one may derive the BIRKOFF's ergodic theorem from a general solution of the restricted ergodic problem. Thus we get a joint solution of the classical and the restricted ergodic problems.

2. - The general ergodic problem.

Given a probability space $(\Omega, \mathscr{F}, m)$ and a measurable space $(\Gamma, \mathscr{G})$, where $\mathscr{G}$ is a σ-algebra of subsets of Γ, consider a measurable transformation S of $(\Omega, \mathscr{F})$ into $(\Gamma, \mathscr{G})$ (see [4], p. 162).

Let M be the measure induced on $\mathscr{G}$ by the measurable transformation S; thus, for each $G \in \mathscr{G}$, is $M(G) = m(S^{-1} G)$.

Moreover, let F be the collection of random variables defined on $(\Gamma, \mathscr{G}, M)$, i.e. the collection of measurable mappings of $(\Gamma, \mathscr{G})$ into $(R_1, \mathscr{B})$.

B. FORTE (see [6]) has derived a necessary and sufficient condition for ergodicity with respect to a given measurable transformation T of Ω onto Ω, of all random variables of F, when the measurable transformation S is defined by a given random variable $f(\omega)$. This result can be extended to the case of an arbitrary measurable transformation S of $(\Omega, \mathscr{F})$ into $(\Gamma, \mathscr{G})$, i.e. we can given a necessary and sufficient condition for all random variables of F to be ergodic with respect to a given measurable transformation T of Ω onto Ω.

In fact the transformation T induces, by the given measurable transformation S, a transformation U_S of $S\Omega$ into Γ. This transformation U_S is defined, for each subset C of $S\Omega$, by the relation:

$$U_S\, C = S\, T\, S^{-1} C\,.$$

Let U_S be defined on the whole Γ and for each subset C of $\Gamma - S\Omega$:

$$U_S\, C = C\,.$$

For each subset C of Γ is thus:

$$U_S\, C = S\, T\, S^{-1}\, C \cap S\Omega + C \cap (\Gamma - S\Omega)\,.$$

A set C of Γ is invariant under the transformation U_S if:

$$U_S^{-1}\, C = C\,.$$

In particular, both sets $S\Omega$ and $\Gamma - S\Omega$, as well as each set $C \subset \Gamma - S\Omega$, are invariant. Given a set $V \in \mathscr{F}$, invariant with respect to the given measurable transformation T, i.e. such that $T^{-1}\, V = V$, let M' be the measure defined on the σ-algebra $\mathscr{G}$ of the subsets G of Γ by the following relation:

$$M'(G) = m(V \cap S^{-1}\, G)$$

and M'' the measure defined on the same σ-algebra of sets by:

$$M''(G) = m[(\Omega - V) \cap S^{-1}\, G]\,.$$

Theorem. *A necessary and sufficient condition for a function $\varphi \in F$ to be ergodic, i.e.*

$$\hat{\varphi}(\omega) = \text{const}\,. \qquad \text{in } (\Omega, \mathscr{F}, m),$$

is that

$$M'(G)\, M''(\Gamma) = M''(G)\, M'(\Gamma)$$

for each invariant set $V \in \mathscr{F}$, and each $G \in \mathscr{G}$ (1).

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(1) If we consider the group of transformations T_t instead of the group $\{T^n\}$ the condition for ergodicity is the same, where of course an invariant set V is a set for which $T_t^{-1}\, V = V$ for every t.

The proof is contained in [6], where one needs only to change the meaning of the symbols.

We shall say that a measurable transformation S of $(\Omega, \mathscr{F})$ into $(\Gamma, \mathscr{G})$ is ergodic with respect to the transformation T if it satisfies the condition of the theorem just mentioned; this is in full analogy with the classical case for a transformation T which satisfies the condition of the BIRKOFF's theorem.

We shall prove that, if we consider the particular transformation $S \equiv T$ of $(\Omega, \mathscr{F})$ onto $(\Omega, \mathscr{F})$, the established condition for a random variable $\varphi \in F$ to be ergodic is the same as the classical condition for a transformation T to be ergodic. The classical condition for ergodicity is therefore a particular case of the given condition, if we of course identify the general transformation S with the measurable transformation T of $(\Omega, \mathscr{F})$ onto $(\Omega, \mathscr{F})$.

Clearly, if the transformation T is ergodic in the sense of BIRKOFF and V is an invariant set for it, then necessarily $m(V) = 0$ or $m(V) = 1$. Thus we have, for any decomposition of Ω into two invariant disjointed sets V', V'',

$$m(V') = 0, \qquad m(V'') = 1 .$$

Let G be any measurable subset of Ω and M', M'' the measures associated to G as mentioned above. It is:

$$M'(G) = m(V' \cap T^{-1} G),$$

$$M''(G) = m(V'' \cap T^{-1} G) = m[(\Omega - V') \cap T^{-1} G] .$$

Therefore

$$M'(G) = 0, \qquad M''(G) = m(T^{-1} G) ,$$

and similarly

$$M'(\Omega) = m(V' \cap T^{-1} \Omega) = 0, \qquad M''(\Omega) = m(V'' \cap T^{-1} \Omega) = 1 .$$

It follows

$$M'(G)\, M''(\Omega) = 0, \qquad M''(G)\, M'(\Omega) = 0 ,$$

so that

$$M'(G)\, M''(\Omega) = M''(G)\, M'(\Omega)$$

for each $G \in \mathscr{F}$ and each choice of the invariant set V' in Ω.

Thus, if T is ergodic in the sense of BIRKOFF, T is ergodic also with respect to itself.

Conversely, suppose that for each $G \in \mathscr{F}$ and every invariant set V' of Ω is:

$$M'(G)\, M''(\Omega) = M''(G)\, M'(\Omega).$$

We shall prove that either $m(V') = 0$ or $m(V') = 1$. Thereafter we shall conclude that if the transformation T is ergodic with respect to itself, it is ergodic in the sense of BIRKOFF.

In fact, let exist in Ω a set V', invariant under the transformation T and such that $m(V') \neq 0$ and $m(\Omega - V') \neq 0$. Consider the set $G \equiv V'$. It is clear that:

$$M'(G) = m(V' \cap T^{-1} V') = m(V'),$$

$$M''(G) = m[(\Omega - V') \cap T^{-1} V'] = 0\,,$$

$$M'(\Omega) = m(V' \cap T^{-1} \Omega) = m(V'),$$

$$M''(\Omega) = m[(\Omega - V') \cap T^{-1} \Omega] = m(\Omega - V')\,.$$

Therefore:

$$M'(G)\, M''(\Omega) = m(V')\, m(\Omega - V') \neq 0\,,$$

$$M''(G)\, M'(\Omega) = 0\,,$$

i.e. the transformation T is not ergodic with respect to itself.

From this it follows the full equivalence of the two conditions for ergodicity, with $S \equiv T$.

3. - Intrinsic character of the ergodicity's condition for Hamiltonian systems.

Consider two measure-spaces $(\Omega, \mathscr{F}, m)$ and $(\overline{\Omega}, \overline{\mathscr{F}}, \overline{m})$.

Let $\overline{\Omega}$ be, as well as Ω, a subset of R^{2n} and $\overline{\mathscr{F}}$ a σ-algebra of subsets of $\overline{\Omega}$ (for instance the σ-algebra of the L-measurable subsets of $\overline{\Omega}$, if such is $\overline{\Omega}$); let $\overline{m}$ be a probability measure defined on the σ-algebra $\overline{\mathscr{F}}$.

Let R be a one to one measurable transformation of Ω onto $\overline{\Omega}$, such that to every set $A \in \mathscr{F}$ corresponds a set $\overline{A} \in \overline{\mathscr{F}}$. In particular $R\Omega = \overline{\Omega}$.

Suppose further that R is measure-preserving, i.e. for every set $\overline{A}$

$$m\, R^{-1}\, \overline{A} = \overline{m}\, \overline{A}$$

($m\, R^{-1}$ is the measure induced by m on the σ-algebra $\overline{\mathscr{F}}$).

The measurable transformation T of Ω onto Ω induces, by the measurable transformation R of Ω onto $\overline{\Omega}$, a transformation $\overline{T}$ of $\overline{\Omega}$ onto $\overline{\Omega}$. This transformation $\overline{T}$ is defined, for each set $\overline{A} \in \overline{\mathscr{F}}$, by the relation

$$\overline{T}\, \overline{A} = R\, T\, R^{-1}\, \overline{A}$$

which is clearly measurable, as a product of measurable transformations. It preserves moreover the measure in $\overline{\Omega}$. We have indeed:

$$\overline{m}(\overline{T}\,\overline{A}) = m(R^{-1}\,\overline{T}\,\overline{A}) = m(T\,R^{-1}\,\overline{A}) = m(R^{-1}\,\overline{A}) = \overline{m}\,\overline{A}\,.$$

The transformation R has further the following properties: every set $V \in \mathscr{F}$ invariant under the transformation T, i.e. such that $T^{-1}\,V = V$ has an image $R\,V = \overline{V} \in \overline{\mathscr{F}}$, invariant under the transformation $\overline{T}$, i.e. a set $\overline{V} \in \overline{\mathscr{F}}$, such that $\overline{T}^{-1}\,\overline{V} = \overline{V}$; conversely every set $\overline{V} \in \overline{\mathscr{F}}$ invariant under the transformation $\overline{T}$ ($\overline{T}^{-1}\,\overline{V} = \overline{V}$) has as inverse image $V = R^{-1}\,\overline{V}$ invariant under the transformation T ($T^{-1}\,V = V$).

Let S be a measurable transformation of $(\Omega, \mathscr{F})$ into the measurable space $(\Gamma, \mathscr{G})$ ($\mathscr{G}$, as usual, is a σ-algebra of subsets of Γ). By the transformation R of $(\Omega, \mathscr{F})$ onto $(\overline{\Omega}, \overline{\mathscr{F}})$ we get a transformation $S\,R^{-1}$ of $\overline{\Omega}$ into Γ. Given any set $G \in \mathscr{G}$ we have $S^{-1}\,G \in \mathscr{F}$ and $(R^{-1})^{-1}\,S^{-1}\,G = R\,S^{-1}\,G \in \overline{\mathscr{F}}$. The transformation $S\,R^{-1}$ of $\overline{\Omega}$ into Γ is therefore a measurable transformation of $(,\overline{\Omega}\;\overline{\mathscr{F}})$ into $(\Gamma, \mathscr{G})$. One can prove that, if the measurable transformation R is ergodic with respect to the transformation T, then the measurable transformation $S\,R^{-1}$ is ergodic with respect to the transformation $\overline{T}$.

Suppose indeed that the transformation R is ergodic with respect to the transformation T. The conditions of the above mentioned theorem are then satisfied and we can say: if V is an arbitrary set of $\mathscr{F}$, invariant under the measurable transformation T, define the measures M' and M'' on the σ-algebra of the subsets $G \in \mathscr{G}$ by

$$M'(G) = m(V \cap S^{-1}\,G), \qquad M''(G) = m[(\Omega - V) \cap S^{-1}\,G]\,.$$

If follows

$$M'(G)\;M''(\Gamma) = M''(G)\;M'(\Gamma).$$

We shall prove that the transformation $S\,R^{-1}$ is ergodic under the transformation $\overline{T}$, by verifying that if $\overline{V}$ is a set of $\overline{\mathscr{F}}$, invariant under the measurable transformation $\overline{T}$ and $\overline{M}'$, $\overline{M}''$ are the measures defined on the σ-algebra $\mathscr{G}$ of the subsets G of Γ by

$$\overline{M}'(G) = \overline{m}(\overline{V} \cap R\,S^{-1}\,G), \qquad \overline{M}''(G) = \overline{m}[(\overline{\Omega} - \overline{V}) \cap R\,S^{-1}\,G],$$

then

$$\overline{M}'(G)\;\overline{M}''(\Gamma) = \overline{M}''(G)\;\overline{M}'(\Gamma).$$

In fact the inverse image of any set $\overline{V} \in \overline{\mathscr{F}}$ invariant under the transformation $\overline{T}$ is, as we have seen, a set $V \in \mathscr{F}$ invariant under the transformation T; moreover, since the transformation R is one to one, for every set $G \in \mathscr{G}$ we have

$$\overline{V} \cap R\, S^{-1}\, G = R\, V \cap R\, S^{-1}\, G = R(V \cap S^{-1}\, G)\,.$$

Since the transformation R is measure-preserving, it follows

$$\overline{m}(\overline{V} \cap R\, S^{-1}\, G) = \overline{m}[R(V \cap S^{-1}\, G)] = m(V \cap S^{-1}\, G).$$

The first term of these equalities is, by definition, $\overline{M}'(G)$ and the last one is $M'(G)$. If follows that, for every $G \in \mathscr{G}$

$$\overline{M}'(G) = M'(G).$$

In particular for $G \equiv \Gamma$ we get

$$\overline{M}'(\Gamma) = M'(\Gamma).$$

Similarly, for the difference $\Omega - V$ and the corresponding difference $\overline{\Omega} - \overline{V}$ we have

$$\overline{m}[(\overline{\Omega} - \overline{V}) \cap R\, S^{-1}\, G] = m[(\Omega - V) \cap S^{-1}\, G]\,,$$

therefore

$$\overline{M}''(G) = M''(G) \qquad \text{for each } G \in \mathscr{G}.$$

In particular for $G \equiv \Gamma$ we get

$$\overline{M}''(\Gamma) = M''(\Gamma).$$

Thus, if the transformation S is ergodic with respect to the transformation T, i.e. if for every invariant set V and for every set $G \in \mathscr{G}$ we have

$$M'(G)\, M''(\Gamma) = M''(G)\, M'(\Gamma),$$

we shall also have

$$\overline{M}'(G)\, \overline{M}''(\Gamma) = \overline{M}''(G)\, \overline{M}'(\Gamma)\,.$$

This states that the transformation $R^{-1}\, S$ is also ergodic with respect to the transformation $\overline{T}$.

Since the canonical transformations of R^{2n} onto R^{2n} satisfie the hypotheses for the preceding result to be valid, we derive that the ergodicity condition mentioned above is invariant under such transformations. Therefore this condition of ergodicity states an intrinsic property for every dynamical system.

4. - Conclusion.

All that has been developped in the preceding paragraphs about the condition of ergodicity resolve both the restricted and the classical ergodic problems. We believe that this condition may be very useful in finding those dynamical systems for which given microscopic state functions are ergodic.

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Summary.

On the basis of a solution of the restricted ergodic problem, we show that this problem contains the classical ergodic problem and that they are solved by the same condition of ergodicity. We demonstrate moreover that this condition of ergodicity is intrinsic for every given dynamical system.

Riassunto.

Sulla base di un risultato precedente, che risolve il problema ergodico ristretto, si riconosce che tale problema contiene il problema ergodico classico ed è risolto dalla stessa condizione dt ergodicità. Si dimostra altresì che tale condizione di ergodicità è intrinseca per ogni dato sistema dinamico.

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