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H. N. NIGAM (*)

Use of the Generalized Laplace Transform to Integral Functions of Several Complex Variables. (**)

1. – Let

$$(1.1)\quad F(z) = F(z_1, \ldots, z_n) = \sum_{k_1, \ldots, k_n = 0}^{\infty} a_{k_1, \ldots, k_n} \frac{(k_1+k_2+1) \ldots (k_1+\ldots+k_{n-1}+1)}{k_1! \ldots k_n!} z_1^{k_1} \ldots z_n^{k_n}$$

be an integral function of n-complex variables $z_1, \ldots, z_n$. Denote

$$M_{G; \varrho_1, \ldots, \varrho_n}(r, F) = \max_{(z_1, \ldots, z_n) \in G} | F(r^{\varrho_1} z_1, \ldots, r^{\varrho_n} z_n) | ,$$

where G be the closed polycircular domain in the space $z = (z_1, \ldots, z_n)$ and $\varrho_1, \ldots, \varrho_n$ being the positive numbers, then according to A. A. GOLDBERG [1]:

The integral function $F(z_1, \ldots, z_n)$ will be called $(G; \varrho_1, \ldots, \varrho_n)$-*order* and $(G; \varrho_1, \ldots, \varrho_n)$-*type* respectively, if

$$\limsup_{r\to\infty} \left\{ \frac{1}{\log r} \log \log M_{G; \varrho_1, \ldots, \varrho_n}(r, F) \right\} = \varrho$$

and

$$\limsup_{r\to\infty} \left\{ r^{-\varrho} \log M_{G; \varrho_1, \ldots, \varrho_n}(r, F) \right\} = \sigma .$$

(*) Indirizzo: Department of Mathematics and Astronomy, Lucknow University, Lucknow, India.

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MEIJER [2] has given a generalisation of the classical LAPLACE transform:

$$\varphi(p) = p \int_0^\infty e^{-pt}\, g(t)\, dt, \qquad \operatorname{Re} p > 0, \tag{1.2}$$

by means of the integral equation

$$\varphi(p) = p \int_0^\infty e^{-pt/2}\, (pt)^{-k-(1/2)}\, W_{k+(1/2),\, m}(pt)\, g(t)\, dt, \qquad \operatorname{Re} p > 0 . \tag{1.3}$$

We shall denote (1.2) and (1.3) symbolically by

$$\varphi(p) = L[g(t)] \quad \text{and} \quad \varphi(p) = M[g(t);\, k + (1/2),\, m]$$

respectively.

The object of this paper is to obtain a new type of relationship between the integral function $F(z_1, \ldots, z_n)$ and the associate function by the help of MEIJER transform, on taking the $(G;\, \varrho_1, \ldots, \varrho_n)$*-order* to be one.

2. – Theorem 1. *Let*

$$F(z_1, \ldots, z_n) = \sum_{k_1, \ldots, k_n = 0}^{\infty} a_{k_1, \ldots, k_n} \frac{(k_1 + k_2 + 1) \ldots (k_1 + \ldots + k_{n-1} + 1)}{k_1! \ldots k_n!}\, z_1^{k_1} \ldots z_n^{k_n} \tag{2.1}$$

be an integral function of n-complex variables $z_1, \ldots, z_n$ satisfying

$$\limsup_{r \to \infty} \left\{ r^{-1} \log M_{G;\, \varrho_1, \ldots, \varrho_n}(r,\, F) \right\} \leqslant \sigma , \tag{2.2}$$

and let

$$f(z_1, \ldots, z_n) = \sum_{k_1, \ldots, k_n = 0}^{\infty} \frac{a_{k_1, \ldots, k_n}}{(k_1 + \ldots + k_n + 1)!}\, b_{k_1, \ldots, k_n}\, z_1^{-(k_1+1)} \ldots z_n^{-(k_n+1)} \tag{2.3}$$

where (1)

$$b_{k_1, \ldots, k_n} = p^{m+(1/2)}\, (a + p)^{-(\nu + k_1 + \ldots + k_n + m + 1/2)} \frac{\Gamma_\times(\nu + k_1 + \ldots + k_n + \frac{1}{2} \pm m)}{\Gamma(\nu + k_1 + \ldots + k_n + \frac{1}{2} - k)} \,\cdot$$

$$\cdot\, {}_2F_1\left[\begin{matrix} \nu + k_1 + \ldots + k_n + \frac{1}{2} + m, \quad m - k \\ \nu + k_1 + \ldots + k_n + \frac{1}{2} - k \end{matrix} \;;\; \frac{a}{a+p}\right], \qquad \operatorname{Re}(\nu + \tfrac{1}{2} \pm m) > 0,$$

(1) The symbol $\Gamma_\times(a \pm b)$ is used to denote $\Gamma(a+b)\,\Gamma(a-b)$ and

$${}_2F_1\left[\begin{matrix} a \pm b \\ c \end{matrix} \;;\; z\right] \quad \text{to denote} \quad {}_2F_1\left[\begin{matrix} a + b,\; a - b \\ c \end{matrix} \;;\; z\right].$$

be the function associated with $F(z_1, \ldots, z_n)$ *and is regular for* $|z_j| > \sigma$ $(j = 1, \ldots, n)$; *then*

$$(2.4) \qquad f(z_1, \ldots, z_n) = \int_0^\infty \ldots \int_0^\infty (z_1 t_1 + \ldots + z_{n-1} t_{n-1})^{-1} \ldots (z_1 t_1 + z_2 t_2)^{-1} \cdot$$

$$\cdot H_{p,a}(z_1 t_1 + \ldots + z_n t_n) F(t_1, \ldots, t_n) \, dt_1 \ldots dt_n,$$

where

$$H_{p,a}(z_1 t_1 + \ldots + z_n t_n) =$$

$$= e^{-\{(a/p)+(1/2)\} p(z_1 t_1 + \ldots + z_n t_n)} W_{k+(1/2),m} [p(z_1 t_1 + \ldots + z_n t_n)] (z_1 t_1 + \ldots + z_n t_n)^{\nu-2},$$

provided the change of order of integration and summation is justified and the series involved converges uniformly and absolutely.

Proof.

Let $F(z_1, \ldots, z_n)$ be an integral function and satisfies (2.2). Then for $\operatorname{Re} z_j = x_j > \sigma > 0$ $(j = 1, \ldots, n)$, we have

$$I_{k_1,\ldots,k_n}(p, a) = \int_0^\infty \ldots \int_0^\infty e^{-\{(a/p)+(1/2)\} p(z_1 t_1 + \ldots + z_n t_n)} W_{k+(1/2),m} [p(z_1 t_1 + \ldots + z_n t_n)] \cdot$$

$$\cdot (z_1 t_1 + \ldots + z_{n-1} t_{n-1})^{-1} \ldots (z_1 t_1 + z_2 t_2)^{-1} (z_1 t_1 + \ldots + z_n t_n)^{\nu-2} F(t_1, \ldots, t_n) \, dt_1 \ldots dt_n =$$

$$= \sum_{k_1,\ldots,k_n=0}^{\infty} a_{k_1,\ldots,k_n} \frac{(k_1+k_2+1) \ldots (k_1+\ldots+k_{n-1}+1)}{k_1! \ldots k_n!} \int_0^\infty \ldots \int_0^\infty e^{-\{(a/p)+(1/2)\} p(z_1 t_1 + \ldots + z_n t_n)} \cdot$$

$$(2.5) \qquad \cdot W_{k+(1/2),m} [p(z_1 t_1 + \ldots + z_n t_n)] (z_1 t_1 + \ldots + z_{n-1} t_{n-1})^{-1} \ldots (z_1 t_1 + z_2 t_2)^{-1} \cdot$$

$$\cdot (z_1 t_1 + \ldots + z_n t_n)^{\nu-2} t_1^{k_1} \ldots t_n^{k_n} \, dt_1 \ldots dt_n$$

$$= \sum_{k_1,\ldots,k_n=0}^{\infty} a_{k_1,\ldots,k_n} \frac{(k_1 + k_2 + 1) \ldots (k_1 + \ldots + k_{n-1} + 1)}{k_1! \ldots k_n!} z_1^{-(k_1+1)} \ldots z_n^{-(k_n+1)} \cdot$$

$$\cdot \int_0^\infty \ldots \int_0^\infty e^{-\{(a/p)+(1/2)\} p(\zeta_1 + \ldots + \zeta_n)} W_{k+(1/2),m} [p(\zeta_1 + \ldots + \zeta_n)] \cdot$$

$$\cdot (\zeta_1 + \ldots + \zeta_{n-1})^{-1} \ldots (\zeta_1 + \zeta_2)^{-1} (\zeta_1 + \ldots + \zeta_n)^{\nu-2} \zeta_1^{k_1} \ldots \zeta_n^{k_n} \, d\zeta_1 \ldots d\zeta_n .$$

Regarding the change of order of integration and summation in (2.5), if we replace $a_{k_1,\dots,k_n}$ by $|a_{k_1,\dots,k_n}|$ and

$$(z_1 t_1 + \dots + z_{n-1} t_{n-1})^{-1} \dots (z_1 t_1 + z_2 t_2)^{-1} H_{p,a}(z_1 t_1 + \dots + z_n t_n)$$

by

$$|(z_1 t_1 + \dots + z_{n-1} t_{n-1})^{-1} \dots (z_1 t_1 + z_2 t_2)^{-1} H_{p,a}(z_1 t_1 + \dots + z_n t_n)|,$$

$\operatorname{Re}(z_j) = x_j > \sigma > 0$ $(j = 1, \dots, n)$, and noting that

$$W_{k+(1/2),m}(z) = O(|z|^{\pm m+(1/2)}), \qquad \text{for } |z| \text{ small}$$

and

$$W_{k+(1/2),m}(z) = O(|z|^{k+(1/2)} e^{-1/(2\operatorname{Re} z)}), \qquad \text{for } |z| \text{ large},$$

then, for $\operatorname{Re} p > 0$, $\operatorname{Re} a > 0$, the resulting series converges uniformly and all the terms are positive. Hence the change of order of integration and summation is justified and $f(z_1, \dots, z_n)$ is regular for $|z_j| > \sigma$ $(j = 1, \dots, n)$ and $\operatorname{Re}(\nu + \frac{1}{2} \pm m) > 0$, $\operatorname{Re} p > 0$, $\operatorname{Re} a > 0$.

Let us first prove the above theorem for the case when the integral function is of two variables.

So when

$$\zeta_1 + \zeta_2 = u_1, \quad \zeta_2 = u_1 u_2, \qquad (0 \leqslant u_2 < 1, \quad 0 \leqslant u_1 < +\infty),$$

we have

$$I_{k_1,k_2}(p, a) =$$

$$= \sum_{k_1,k_2=0}^{\infty} \frac{a_{k_1,k_2}}{k_1!\,k_2!} z_1^{-(k_1+1)} z_2^{-(k_2+1)} \int_0^{\infty}\int_0^{\infty} e^{-\{(a/p)+(1/2)\}p(\zeta_1+\zeta_2)} (\zeta_1+\zeta_2)^{\nu-2} W_{k+(1/2),m}[p(\zeta_1+\zeta_2)] \cdot$$
$$\cdot \zeta_1^{k_1} \zeta_2^{k_2} \, d\zeta_1 \, d\zeta_2$$

$$= \sum_{k_1,k_2=0}^{\infty} \frac{a_{k_1,k_2}}{k_1!\,k_2!} z_1^{-(k_1+1)} z_2^{-(k_2+1)} \int_0^{\infty}\int_0^{1} e^{-\{(a/p)+(1/2)\}p u_1} u_1^{k_1+k_2+\nu-1} W_{k+(1/2),m}(p u_1) u_2^{k_2} \cdot$$
$$\cdot (1-u_2)^{k_1} \, du_1 \, du_2 .$$

Evaluating u_2-integral with the help of the Eulerian-integral of the first kind [3], and making a simple transformation, we can replace the double integral by

$$\frac{k_1!\,k_2!}{(k_1+k_2+1)!} p^{-(\nu+k_1+k_2)} \int_0^{\infty} e^{-\{(a/p)+(1/2)\}} x^{\nu+k_1+k_2-1} W_{k+(1/2),m}(x) \, dx .$$

Now evaluating the x-integral with the help of GOLDSTEIN's integral ([4], p. 114), we get

$$I_{k_1,k_2}(p,\, a) = \sum_{k_1,k_2=0}^{\infty} \frac{a_{k_1,k_2}}{(k_1+k_2+1)!}\, p^{m+\frac{1}{2}}\, (a+p)^{-(\nu+k_1+k_2+m+\frac{1}{2})}\, \frac{\Gamma \times (\nu+k_1+k_2+\frac{1}{2} \pm m)}{\Gamma(\nu+k_1+k_2+\frac{1}{2}-k)} \,\cdot$$

$$\cdot\; {}_2F_1\left[\begin{matrix}\nu+k_1+k_2+\frac{1}{2}+m,\; m-k \\ \nu+k_1+k_2+\frac{1}{2}-k\end{matrix};\; \frac{a}{a+p}\right] z_1^{-(k_1+1)}\, z_2^{-(k_2+1)},$$

$$\operatorname{Re}(\nu+\tfrac{1}{2}\pm m)>0, \qquad \operatorname{Re}(p+a)>0\,.$$

This shows that the theorem is true for two variables.

We next prove the above theorem for integral function of three and four variables.

Let

$$\zeta_1+\zeta_2+\zeta_3=u_1, \qquad \zeta_1+\zeta_2=u_1u_2, \qquad \zeta_2=u_1u_2u_3;$$

we obtain

$$I_{k_1,k_2,k_3}(p,\, a) = \sum_{k_1,k_2,k_3=0}^{\infty} a_{k_1,k_2,k_3}\, \frac{k_1+k_2+1}{k_1!\,k_2!\,k_3!}\, z_1^{-(k_1+1)}\, z_2^{-(k_2+1)}\, z_3^{-(k_3+1)}\,\cdot$$

$$\cdot \int_0^\infty\!\!\int_0^\infty\!\!\int_0^\infty e^{-\{(a/p)+(1/2)\}p(\zeta_1+\zeta_2+\zeta_3)}\,(\zeta_1+\zeta_2)^{-1}\,(\zeta_1+\zeta_2+\zeta_3)^{\nu-2}\, W_{k+\frac{1}{2},m}\,[p\,(\zeta_1+\zeta_2+\zeta_3)]\,\cdot$$

$$\cdot\; \zeta_1^{k_1}\,\zeta_2^{k_2}\,\zeta_3^{k_3}\,d\zeta_1\,d\zeta_2\,d\zeta_3$$

$$= \sum_{k_1,k_2,k_3=0}^{\infty} a_{k_1,k_2,k_3}\, \frac{k_1+k_2+1}{k_1!\,k_2!\,k_3!}\, z_1^{-(k_1+1)}\, z_2^{-(k_2+1)}\, z_3^{-(k_3+1)}\,\cdot$$

$$\cdot \int_0^\infty\!\!\int_0^1\!\!\int_0^1 e^{-\{(a/p)+(1/2)pu_1\}}\, u_1^{k_1+k_2+k_3+\nu-1} W_{k+\frac{1}{2},m}(p\,u_1)\, u_2^{k_1+k_2}(1-u_2)^{k_3}\, u_3^{k_2}\,(1-u_3)^{k_1} du_1\, du_2\, du_3$$

$$= \sum_{k_1,k_2,k_3=0}^{\infty} \frac{a_{k_1,k_2,k_3}}{(k_1+k_2+k_3+1)!}\, p^{m+\frac{1}{2}}\, (a+p)^{-(\nu+k_1+k_2+k_3+m+\frac{1}{2})}\, \frac{\Gamma \times (\nu+k_1+k_2+k_3+\frac{1}{2}\pm m)}{\Gamma(\nu+k_1+k_2+k_3+\frac{1}{2}-k)}\,\cdot$$

$$\cdot\; {}_2F_1\left[\begin{matrix}\nu+k_1+k_2+k_3+\frac{1}{2}+m,\; m-k \\ \nu+k_1+k_2+k_3+\frac{1}{2}-k\end{matrix};\; \frac{a}{a+p}\right] z_1^{-(k_1+1)}\, z_2^{-(k_2+1)}\, z_3^{-(k_3+1)},$$

$$\operatorname{Re}(\nu+\tfrac{1}{2}\pm m)>0, \qquad \operatorname{Re}(a+p)>0\,.$$

This shows that the theorem is true in case of three variables as well.

Further, in the case of four variables, if we put

$$\zeta_1+\zeta_2+\zeta_3+\zeta_4=u_1, \quad \zeta_1+\zeta_2+\zeta_3=u_1u_2, \quad \zeta_1+\zeta_2=u_1u_2u_3, \quad \zeta_2=u_1u_2u_3u_4,$$

and proceed as in the case of three variables, we get

$$I_{k_1, k_2, k_3, k_4}(p, a) = \sum_{k_1, k_2, k_3, k_4 = 0}^{\infty} \frac{a_{k_1, k_2, k_3, k_4}}{(k_1+k_2+k_3+k_4+1)!} \cdot$$

$$\cdot\, p^{m+1/2}\ (a+p)^{-(\nu+k_1+k_2+k_3+k_4+m+1/2)} \frac{\Gamma_{\times}(\nu + k_1 + k_2 + k_3 + k_4 + \frac{1}{2} \pm m)}{\Gamma(\nu + k_1 + k_2 + k_3 + k_4 + \frac{1}{2} - k)} \cdot$$

$$\cdot\, {}_2F_1\left[\begin{matrix}\nu+k_1+k_2+k_3+k_4+\frac{1}{2}+m,\ m-k \\ \nu+k_1+k_2+k_3+k_4+\frac{1}{2}-k\end{matrix};\ \frac{a}{a+p}\right] z_1^{-(k_1+1)}\, z_2^{-(k_2+1)}\, z_3^{-(k_3+1)}\, z_4^{-(k_4+1)},$$

$$\mathrm{Re}\,(\nu + \tfrac{1}{2} \pm m) > 0, \qquad \mathrm{Re}\,(p + a) > 0,$$

which shows that the theorem is also true for the case of four variables.

Similarly, we can deduce the result in the case of an integral function of n-complex variables.

Corollary. *If we take $a = 0$, then under the conditions of the theorem:*

$$f(z_1, \dots, z_n) = \sum_{k_1, \dots, k_n = 0}^{\infty} \frac{a_{k_1, \dots, k_n}}{(k_1 + \dots + k_n + 1)!} C_{k_1, \dots, k_n} z_1^{-(k_1+1)} \dots z_n^{-(k_n+1)},$$

where

$$C_{k_1, \dots, k_n} = p^{-(\nu + k_1 + \dots + k_n)} \frac{\Gamma_{\times}(\nu+k_1+\dots+k_n+\frac{1}{2}\pm m)}{\Gamma(\nu+k_1+\dots+k_n+\frac{1}{2} - k)},$$

$$\mathrm{Re}\,(\nu + \tfrac{1}{2} \pm m) > 0, \qquad \mathrm{Re}\, p > 0$$

be the function associated with $F(z_1, \dots, z_n)$ and

$$f(z_1, \dots, z_n) =$$

$$= \int_0^{\infty} \cdot \int_0^{\infty} (z_1 t_1 + \dots + z_{n-1} t_{n-1})^{-1} \dots (z_1 t_1 + z_2 t_2)^{-1} H_p(z_1 t_1 + \dots + z_n t_n)\, F(t_1, \dots, t_n)\, dt_1 \dots dt_n,$$

where

$$H_p(z_1 t_1 + \dots + z_n t_n) =$$

$$= e^{-1/2\, p(z_1 t_1 + \dots + z_n t_n)}\ W_{k+1/2, m}[p(z_1 t_1 + \dots + z_n t_n)]\, (z_1 t_1 + \dots + z_n t_n)^{\nu - 2}.$$

3. – Theorem 2. *Let*

$$(3.1)\qquad F(z_1, \dots, z_n) = \sum_{k_1, \dots k_n = 0}^{\infty} a_{k_1, \dots, k_n} \frac{(k_1+k_2+1) \dots (k_1+\dots+k_{n-1}+1)}{k_1! \dots k_n!} z_1^{k_1} \dots z_n^{k_n}$$

be an integral function of n-complex variables $z_1, \dots, z_n$ *satisfying*

$$(3.2)\qquad \limsup_{r \to \infty} \{ r^{-1} \log M_{G; \varrho_1, \dots, \varrho_n}(r, F) \} \leqslant \sigma$$

and let

$$(3.3)\qquad f(z_1, \dots, z_n) = \sum_{k_1, \dots, k_n = 0}^{\infty} \frac{a_{k_1, \dots, k_n}}{(k_1 + \dots + k_n + 1)!} V_{k_1, \dots, k_n} z_1^{-(k_1+1)} \dots z_n^{-(k_n+1)},$$

where

$$V_{k_1, \dots, k_n} = 2\, p^{1/2} a^{k-(k_1+\dots+k_n)} K_{2m}(2p^{1/2} a^{1/2}), \qquad \operatorname{Re} p > 0, \qquad \operatorname{Re} a > 0,$$

be the function associated with $F(z_1, \dots, z_n)$ *and is regular for* $|z_j| > \sigma$ $(j = 1, \dots, n)$; *then*

$$f(z_1, \dots, z_n) = \int_0^\infty \dots \int_0^\infty (z_1 t_1 + \dots + z_{n-1} t_{n-1})^{-1} \dots$$
$$\dots (z_1 t_1 + z_2 t_2)^{-1} S_{p,a}(z_1 t_1 + \dots + z_n t_n) F(t_1, \dots, t_n)\, dt_1 \dots dt_n,$$

where

$$S_{p,a}(z_1 t_1 + \dots + z_n t_n) = e^{-a(z_1 t_1 + \dots + z_n t_n)} e^{-\frac{1}{2} p (z_1 t_1 + \dots + z_n t_n)^{-1}} \cdot$$
$$\cdot (z_1 t_1 + \dots + z_n t_n)^{-k-(3/2)} W_{k-(k_1+\dots+k_n)+\frac{1}{2}, m} \left[p (z_1 t_1 + \dots + z_n t_n)^{-1} \right],$$

provided the change of order of integration and summation is justified and the series involved converges uniformly and absolutely.

Proof. The proof is similar to that of Theorem 1, except that we use GOLDSTEIN's operational representation ([4], p. 107)

$$2\, p^{1/2} a^{k+1/2} K_{2m}(2\, p^{1/2} a^{1/2}) = L[e^{-\frac{1}{2}(p/x)} W_{k,m}(p/x)\, x^{-k}], \qquad \operatorname{Re} p > 0, \quad \operatorname{Re} a > 0;$$

instead of ([4], p. 114).

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