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On the Absolute Nörlund Summability Factors of a Fourier Series. (**)

1.1. - Definitions.

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \dots + p_n, \qquad P_{-1} = p_{-1} = 0 .$$

The sequence-to-sequence transformation:

$$t_n = \sum_{\nu=0}^{n} \frac{p_{n-\nu}\, s_\nu}{P_n} \qquad (P_n \neq 0) \tag{1.1.1}$$

defines the sequence $\{t_n\}$ of NÖRLUND means (1) of the sequence $\{s_n\}$, generated by the sequence of coeffcients $\{p_n\}$.

The series $\sum a_n$ is said to be summable (N, p_n) to the sum s, if $\lim_{n\to\infty} t_n$ exists and is equal to s, and is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation (2), that is to say,

$$\sum_n |t_n - t_{n-1}| \leqslant K \quad (3).$$

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(1) NÖRLUND [10]; substantially the same definition was given by G. F. WORONOI in the Proceedings of the 11th Congress of Russian naturalists and scientists (in Russian), St. Petersburg 1902, pp. 60-61. An English translation of this work of WORONOI with « remarks of the translator » by J. D. TAMARKIN is contained in WORONOI [13].

(2) Symbolically, $\{t_n\} \in BV$; similarly by « $f(x) \in BV(h, k)$ » we shall mean that $f(x)$ is a function of bounded variation over the interval (h, k).

(3) MEARS [8]. K denotes throughout an absolute constant, not necessarily the same at each occurence.

In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\,\Gamma(\alpha)} \qquad (\alpha \geqslant 0), \tag{1.1.2}$$

the NÖRLUND mean reduces to familiar (C, α)-mean [4]. The summability $|N, p_n|$, with p_n defined by (1.1.2), is thus the same as summability $|C, \alpha|$ [5].

Similarly, in the case in which

$$\begin{cases} p_n = 1/(n+1) \qquad (n \geqslant 0) \\ P_n = 1 + (1/2) + \ldots + (1/(n+1)) \sim \log n, \qquad \text{as } n \to \infty, \end{cases} \tag{1.1.3}$$

the NÖRLUND mean reduces to the familiar «harmonic mean» [6], and summability $|N, p_n|$ is then the same as absolute harmonic summability, or simply the summability $|N, 1/(n+1)|$.

It is known that the harmonic mean is both regular [7] and absolutely regular [8]; and the summability $|N, 1/(n+1)|$ implies summability $|C, \alpha|$ for every positive α [9].

1.2. – Let $f(t)$ be a periodic function, with period 2π, and integrable in the sense of LEBESGUE over $(-\pi, \pi)$. We assume, as we may without any loss of generality, that the constant term in the FOURIER series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)\, dt = 0,$$

and

$$f(t) \sim \sum (a_n \cos nt + b_n \sin nt) = \sum A_n(t).$$

We write throughout

$$\Phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\},$$

$$\Phi_\alpha(t) = \frac{\alpha}{t^\alpha} \int_0^t (t-u)^{\alpha-1}\, \Phi(u)\, du \qquad (\alpha > 0), \qquad \Phi_0(t) = \Phi(t),$$

$$t_n^1 = \frac{1}{n} \sum_{\nu=1}^{n} \nu\, a_\nu, \qquad \tau_n^{*1} = \frac{1}{n} \sum_{\nu=1}^{n} \nu\, A_\nu(x).$$

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(4) HARDY [4], § 5.13.

(5) Summability $|C, \alpha|$ was defined by FEKETE [3], and KOGBETLIANTZ [5].

(6) HARDY [4], § 5.13; RIESZ [12].

(7) HARDY [4], § 4.2.

(8) For absolute regularity of NÖRLUND means, see MEARS [9].

(9) MCFADDEN [7].

For any sequence $\{\lambda_n\}$, we write

$$\Delta\lambda_n = \lambda_n - \lambda_{n+1}, \qquad \Delta^2\lambda_n = \Delta(\Delta\lambda_n)\,.$$

A sequence $\{\lambda_n\}$ is said to be convex [10] if

$$\Delta^2\lambda_n \geqslant 0, \qquad n = 0,\ 1, \ldots\,.$$

1.3. – The following theorems on absolute CESÀRO summability factors of a FOURIER-LEBESGUE series are known.

Theorem A [11]. *If $\{\lambda_n\}$ is a convex sequence such that the series $\sum n^{-1}\lambda_n$ is convergent, and $\Phi_\alpha(t) \in \mathrm{BV}(0, \pi)$, $0 \leqslant \alpha \leqslant 1$, then the series $\sum \lambda_n A_n(t)$, at $t = x$, is summable $|\mathrm{C}, \alpha|$.*

Theorem B [12]. *If $\{\lambda_n\}$ is a convex sequence such that the series $\sum n^{-1}\lambda_n$ is convergent, and for $0 < \alpha \leqslant 1$*

$$\int_0^t u\,|\,\mathrm{d}\,\Phi_\alpha(t)\,| = O(t), \qquad 0 \leqslant t \leqslant \pi,$$

then the series $\sum (\log \overline{n+1})^{-1}\,\lambda_n\,A_n(t)$, at $t = x$, is summable $|\mathrm{C}, \alpha|$.

It is known [13] that, if (1) p_n is non-negative and non-increasing and (2) p_{n+1}/p_n is non-decreasing, then $|\mathrm{N}, p_n|$ implies $|\mathrm{C}, 1|$.

The object of the present paper is to improve upon Theorems A and B, in the case in which $\alpha = 1$, by replacing $|\mathrm{C}, 1|$ by $|\mathrm{N}, p_n|$, with p_n more general than that characterized above. The results are embodied in Theorems 2 and 3. We prove these theorems by establishing as Theorem 1, a result on the absolute NÖRLUND summability factors of infinite series in general. In n. **2.5** we deduce a number of corollaries which generalize the following results of LAL on the absolute harmonic summability factors.

Theorem C [14]. *If $t_n^1 = O(1)$, as $n \to \infty$, and $\{\lambda_n\}$ is a convex sequence such that the series $\sum n^{-1}\lambda_n$ is convergent, then the series $\sum ((\log n)/n)\lambda_n\,a_n$ is summable $|\mathrm{N}, 1/(n+1)|$.*

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(10) ZYGMUND [**14**], § **3.5**, p. 58.
(11) PRASAD and BHATT [**11**], Theorem 5.
(12) PRASAD and BHATT [**11**], Theorem 7.
(13) MCFADDEN [**7**], Theorem 2.28.
(14) LAL [**6**], Theorem 2.

Theorem D [15]. *If* $\Phi_1(t) \in \mathrm{BV}(0, \pi)$, *and* $\{\lambda_n\}$ *is a convex sequence such that the series* $\sum n^{-1}\lambda_n$ *is convergent, then the series* $\sum ((\log n)/n)\, \lambda_n\, A_n(t)$, *at* $t = x$, *is summable* $|\mathrm{N}, 1/(n+1)|$.

2.1. – We establish the following theorems.

Theorem 1. *Let* $p_0 > 0$, *and let* p_n *be non-negative and non-increasing. If* $t_n^1 = O(\mu_n)$, *as* $n \to \infty$, *where* $\{\mu_n\}$ *is a positive, non-decreasing sequence and if the sequence* $\{\varepsilon_n\}$ *is such that*

$$\text{(i)} \qquad \sum (\mu_n/P_n)\,|\varepsilon_n| < \infty, \qquad \textit{and} \qquad \text{(ii)} \qquad \sum \mu_n\,|\Delta\varepsilon_n| < \infty,$$

then the series $\sum \varepsilon_n a_n$ *is summable* $|\mathrm{N}, p_n|$.

Theorem 2. *Let* $p_0 > 0$, *and let* p_n *be non-negative and non-increasing. If* $\Phi_1(t) \in \mathrm{BV}(0, \pi)$, *and if the sequence* $\{\varepsilon_n\}$ *is such that*

$$\text{(i)} \qquad \sum |\varepsilon_n|/P_n < \infty, \qquad \textit{and} \qquad \text{(ii)} \qquad \sum |\Delta\varepsilon_n| < \infty,$$

then the series $\sum \varepsilon_n A_n(t)$, *at* $t = x$, *is summable* $|\mathrm{N}, p_n|$.

Theorem 3. *Let* $p_0 > 0$, *and let* p_n *be non-negative and non-increasing. If*

$$\int_0^t u\,|\,\mathrm{d}\,\Phi_1(u)\,| = O(t), \qquad 0 \leqslant t \leqslant \pi,$$

and if the sequence $\{\varepsilon_n\}$ *is such that*

$$\text{(i)} \qquad \sum ((\log n)/P_n)\,|\varepsilon_n| < \infty, \qquad \textit{and} \qquad \text{(ii)} \qquad \sum (\log n)\,|\Delta\varepsilon_n| < \infty,$$

then the series $\sum \varepsilon_n A_n(t)$, *at* $t = x$, *is summable* $|\mathrm{N}, p_n|$.

Since a LEBESGUE indefinite integral is absolutely continuous, $\Phi_1(t) \in \mathrm{BV}$ in every range (δ, π), $\delta > 0$. Thus an interesting consequence of Theorem 2 is that the summability $|\mathrm{N}, p_n|$ (with p_n defined as in Theorem 2) of the series $\sum \varepsilon_n A_n(x)$ is a local property.

2.2. – We require following lemmas for the proof of our theorems.

Lemma 1. *If* $p_0 > 0$, *and* p_n *is non-negative and non-increasing, then, for* $\nu \geqslant 1$,

$$\sum_{n=\nu}^{\infty} \frac{p_n}{P_n\,P_{n-1}}\, p_{n-\nu} \leqslant \frac{K}{\nu}.$$

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[15] LAL [6], Theorem 1.

Proof. We first observe that under our hypothesis,

$$\sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} \leqslant \frac{K}{P_{\nu-1}} . \tag{2.2.1}$$

Now, we have

$$\begin{aligned}\sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} p_{n-\nu} &= \sum_{n=\nu}^{2\nu-1} \frac{p_n}{P_n P_{n-1}} p_{n-\nu} + \sum_{n=2\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} p_{n-\nu} \\ &\leqslant \frac{p_\nu}{P_\nu P_{\nu-1}} \sum_{n=\nu}^{2\nu-1} p_{n-\nu} + p_\nu \sum_{n=2\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} \qquad \text{[by (2.2.1.)]} \\ &\leqslant \frac{p_\nu P_{\nu-1}}{P_\nu P_{\nu-1}} + \frac{K p_\nu}{P_{2\nu-1}} \\ &\leqslant \frac{p_\nu}{P_\nu} + \frac{K p_\nu}{P_\nu} \leqslant \frac{1}{\nu+1} + \frac{K}{\nu+1} \leqslant \frac{K}{\nu},\end{aligned}$$

since $(\nu+1)\, p_\nu \leqslant P_\nu$.

Lemma 2. *If $p_0 > 0$, and p_n is non-negative and non-increasing, then, for $\nu \geqslant 1$,*

$$\sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} (P_n - P_{n-\nu}) \leqslant K .$$

Proof. By hypothesis, we have

$$\sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} (P_n - P_{n-\nu}) \leqslant \nu \sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} p_{n-\nu} \leqslant \nu \frac{K}{\nu} = K ,$$

by Lemma 1.

Lemma 3. *If $p_0 > 0$, and p_n is non-negative and non-increasing, then, for $\nu \geqslant 1$,*

$$\sum_{n=\nu}^{\infty} \frac{|\Delta_n p_{n-\nu-1}|}{P_{n-1}} \leqslant \frac{K}{P_\nu} + \frac{K}{\nu} .$$

Proof. Since p_n is non-increasing, we have

$$\sum = \sum_{n=\nu}^{\infty} \frac{|\Delta_n p_{n-\nu-1}|}{P_{n-1}}$$

$$= \frac{p_0}{P_{\nu-1}} + \sum_{n=\nu+1}^{2\nu-1} \frac{\Delta_n p_{n-\nu-1}}{P_{n-1}} + \sum_{n=2\nu}^{\infty} \frac{\Delta_n p_{n-\nu-1}}{P_{n-1}}.$$

Now,

$$\sum\nolimits_1 \equiv \sum_{n=\nu+1}^{2\nu-1} \frac{\Delta_n p_{n-\nu-1}}{P_{n-1}} \leqslant \frac{1}{P_\nu} \sum_{n=\nu+1}^{2\nu-1} (p_{n-\nu-1} - p_{n-\nu}) = \frac{p_0 - p_{\nu-1}}{P_\nu},$$

so that

$$\left|\sum\nolimits_1\right| \leqslant \frac{p_0}{P_\nu} + \frac{p_{\nu-1}}{P_{\nu-1}} \leqslant \frac{p_0}{P_\nu} + \frac{1}{\nu},$$

since $(\nu + 1)\, p_\nu \leqslant P_\nu$.

And, for every integer $m > 2\nu$, by ABEL's transformation, we have

$$\sum\nolimits_2 \equiv \sum_{n=2\nu}^{m} \frac{\Delta_n p_{n-\nu-1}}{P_{n-1}}$$

$$= \sum_{n=2\nu}^{m-1} \frac{p_n}{P_n P_{n-1}} \sum_{\mu=2\nu}^{n} (p_{\mu-\nu-1} - p_{\mu-\nu}) + \frac{1}{P_{m-1}} \sum_{\mu=2\nu}^{m} (p_{\mu-\nu-1} - p_{\mu-\nu})$$

$$= \sum_{n=2\nu}^{m-1} \frac{p_n}{P_n P_{n-1}} (p_{\nu-1} - p_{n-\nu}) + \frac{p_{\nu-1} - p_{m-\nu}}{P_{m-1}},$$

so that

$$\left|\sum\nolimits_2\right| \leqslant p_{\nu-1} \sum_{n=2\nu}^{m-1} \frac{p_n}{P_n P_{n-1}} + \sum_{n=2\nu}^{m-1} \frac{p_n}{P_n P_{n-1}} p_{n-\nu} + \frac{p_{\nu-1}}{P_{m-1}} + \frac{p_{m-\nu}}{P_{m-1}}$$

$$< \frac{K p_{\nu-1}}{P_{2\nu-1}} + \frac{K}{\nu} + \frac{p_{\nu-1}}{P_{\nu-1}} + \frac{p_{m-\nu}}{P_{m-\nu}} \quad \text{[by (2.2.1)]}$$

$$< \frac{K p_{\nu-1}}{P_{\nu-1}} + \frac{K}{\nu} + \frac{1}{\nu} + \frac{1}{m - \nu + 1}$$

$$< \frac{K}{\nu} + \frac{K}{\nu} + \frac{1}{\nu} + \frac{1}{\nu + 1} \leqslant \frac{K}{\nu}.$$

Therefore

$$\sum \leqslant \frac{K}{P_\nu} + \frac{K}{\nu}\,.$$

This completes the proof of the lemma.

Lemma 4. *If* $p_0 > 0$, *and* p_n *is non-negative and non-increasing, then, for* $\nu \geqslant 1$,

$$\sum_{n=\nu}^{\infty} \frac{p_{n-\nu} - p_n}{P_{n-1}} \leqslant K\,.$$

Proof. By ABEL's transformation, we have, for every integer $m > \nu$,

$$\sum{}' = \sum_{n=\nu}^{m} \frac{p_{n-\nu} - p_n}{P_{n-1}}$$

$$= \sum_{n=\nu}^{m-1} \frac{p_n}{P_n P_{n-1}} \sum_{\mu=\nu}^{n} (p_{\mu-\nu} - p_\mu) + \frac{1}{P_{m-1}} \sum_{\mu=\nu}^{m} (p_{\mu-\nu} - p_\mu)$$

$$= \sum_{n=\nu}^{m-1} \frac{p_n}{P_n P_{n-1}} (P_{n-\nu} - P_n + P_{\nu-1}) + \frac{1}{P_{m-1}} (P_{m-\nu} - P_m + P_{\nu-1})$$

$$= -\sum_{n=\nu}^{m-1} \frac{p_n}{P_n P_{n-1}} (P_n - P_{n-\nu}) + P_{\nu-1} \sum_{n=\nu}^{m-1} \frac{p_n}{P_n P_{n-1}} + \frac{P_{\nu-1}}{P_{m-1}} - \frac{P_{m-1} + p_m}{P_{m-1}} + \frac{P_{m-\nu}}{P_{m-1}}$$

$$= -\sum_{n=\nu}^{m-1} \frac{p_n}{P_n P_{n-1}} (P_n - P_{n-\nu}) + P_{\nu-1} \left(\frac{1}{P_{\nu-1}} - \frac{1}{P_{m-1}}\right) + \frac{P_{\nu-1}}{P_{m-1}} - 1 - \frac{p_m}{P_{m-1}} + \frac{P_{m-\nu}}{P_{m-1}}$$

$$= -\sum_{n-\nu}^{m-1} \frac{p_n}{P_n P_{n-1}} (P_n - P_{n-\nu}) + \frac{P_{m-\nu}}{P_{m-1}} + \frac{p_m}{P_{m-1}}\,.$$

Hence

$$\left|\sum{}'\right| \leqslant \sum_{n=\nu}^{m-1} \frac{p_n}{P_n P_{n-1}} (P_n - P_{n-\nu}) + \frac{P_{m-\nu}}{P_{m-\nu}} + \frac{p_m}{P_{m-1}} \leqslant K + 1 \leqslant K\,,$$

by hypothesis and Lemma 2. Hence the result.

Lemma 5 [16]. *If* $\Phi_1(t) \in \mathrm{BV}(0, \pi)$, *then* $\tau_n^{*1} = O(1)$, *as* $n \to \infty$.

Lemma 6 [17]. *If*

$$\int_0^t u \,|\, \mathrm{d}\, \Phi_1(u) \,| = O(t), \qquad 0 \leqslant t \leqslant \pi ,$$

then $\tau_n^{*1} = O(\log n)$, *as* $n \to \infty$.

Lemma 7. *If* $\{\lambda_n\}$ *is a non-increasing sequence such that the series* $\sum n^{-1}\lambda_n$ *is convergent, then*

$$\text{(i)} \quad \sum \Delta\lambda_n < \infty, \qquad \text{(ii)} \quad \sum (\log \overline{n+1})\, \Delta\lambda_n < \infty .$$

This lemma is known [18].

2.3. – Proof of Theorem 1.

Let τ_n be the nth Nörlund mean of the series $\sum_{\nu=0}^{\infty} \varepsilon_\nu a_\nu$. Then by definition,

$$\tau_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} \sum_{\mu=0}^{\nu} \varepsilon_\nu a_\nu = \frac{1}{P_n} \sum_{\nu=0}^{n} P_{n-\nu}\, \varepsilon_\nu a_\nu ,$$

and hence

$$\tau_n - \tau_{n-1} = \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} (P_n p_{n-\nu} - P_{n-\nu} p_n)\, \varepsilon_\nu a_\nu$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} (P_n - P_{n-\nu})\, \varepsilon_\nu a_\nu + \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} (p_{n-\nu} - p_n)\, \varepsilon_\nu a_\nu$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} \Delta_\nu \left\{ (P_n - P_{n-\nu}) \frac{\varepsilon_\nu}{\nu} \right\} \nu t_\nu^1 + \frac{p_n}{P_n P_{n-1}} \left[(P_n - P_{n-\nu-1}) \frac{\varepsilon_{\nu+1}}{\nu+1} \right]_{\nu=n} n t_n^1 +$$

$$+ \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} \Delta_\nu \left\{ (p_{n-\nu} - p_n) \frac{\varepsilon_\nu}{\nu} \right\} \nu t_\nu^1 + \frac{1}{P_{n-1}} \left[(p_{n-\nu-1} - p_n) \frac{\varepsilon_{\nu+1}}{\nu+1} \right]_{\nu=n} n t_n^1$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} \Delta_\nu \left\{ (P_n - P_{n-\nu}) \frac{\varepsilon_\nu}{\nu} \right\} \nu t_\nu^1 + \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} \Delta_\nu \left\{ (p_{n-\nu} - p_n) \frac{\varepsilon_\nu}{\nu} \right\} \nu t_\nu^1 .$$

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[16] This is the particular case of Lemma 9 of [11], when $\alpha = 1$.

[17] This is the particular case of Lemma 11 of [11], when $\alpha = 1$.

[18] (i) is contained in Ahmad [1], Lemma 8. For (ii), see Daniel [2], the lemma on page 69.

Hence

$$|\tau_n - \tau_{n-1}| \leqslant \frac{K\,p_n}{P_n\,P_{n-1}} \sum_{\nu=1}^{n} \left| \Delta_\nu \left\{ (P_n - P_{n-\nu}) \frac{\varepsilon_\nu}{\nu} \right\} \right| \nu\,\mu_\nu +$$

$$+ \frac{K}{P_{n-1}} \sum_{\nu=1}^{n} \left| \Delta_\nu \left\{ (p_{n-\nu} - p_n) \frac{\varepsilon_\nu}{\nu} \right\} \right| \nu\,\mu_\nu$$

$$= \frac{K\,p_n}{P_n\,P_{n-1}} \sum\nolimits_1 + \frac{K}{P_{n-1}} \sum\nolimits_2, \qquad \text{say}.$$

Therefore, in order that $\sum_n |\tau_n - \tau_{n-1}| \leqslant K$, it is sufficient to show that

(2.3.1) $$\sum_n \frac{p_n}{P_n\,P_{n-1}} \sum\nolimits_1 \leqslant K,$$

(2.3.2) $$\sum_n \frac{1}{P_{n-1}} \sum\nolimits_2 \leqslant K.$$

Proof of (2.3.1).

$$\sum\nolimits_1 = \sum_{\nu=1}^{n} \left| \Delta_\nu \left\{ (P_n - P_{n-\nu}) \frac{\varepsilon_\nu}{\nu} \right\} \right| \nu\,\mu_\nu$$

$$\leqslant \sum_{\nu=1}^{n} (P_n - P_{n-\nu}) \frac{|\varepsilon_\nu|}{\nu} \mu_\nu + \sum_{\nu=1}^{n} (P_n - P_{n-\nu}) \,|\Delta \varepsilon_\nu|\, \mu_\nu + \sum_{\nu=1}^{n} p_{n-\nu} \,|\varepsilon_{\nu+1}|\, \mu_\nu$$

$$= \sum\nolimits_{11} + \sum\nolimits_{12} + \sum\nolimits_{13}, \qquad \text{say}.$$

Now,

$$\sum_n \frac{p_n}{P_n\,P_{n-1}} \sum\nolimits_{11} = \sum_{n=1}^{\infty} \frac{p_n}{P_n\,P_{n-1}} \sum_{\nu=1}^{n} (P_n - P_{n-\nu}) \frac{|\varepsilon_\nu|}{\nu} \mu_\nu$$

$$= \sum_{\nu=1}^{\infty} \mu_\nu \frac{|\varepsilon_\nu|}{\nu} \sum_{n=\nu}^{\infty} \frac{p_n}{P_n\,P_{n-1}} (P_n - P_{n-\nu})$$

$$\leqslant K \sum_{\nu=1}^{\infty} \nu^{-1} \mu_\nu \,|\varepsilon_\nu| \qquad \text{(by Lemma 2)}$$

$$\leqslant K, \qquad \text{by hypothesis.}$$

Next,

$$\sum_n \frac{p_n}{P_n P_{n-1}} \sum\nolimits_{12} = \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} (P_n - P_{n-\nu}) \mid \Delta \varepsilon_\nu \mid \mu_\nu$$

$$= \sum_{\nu=1}^{\infty} \mu_\nu \mid \Delta \varepsilon_\nu \mid \sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} (P_n - P_{n-\nu})$$

$$\leqslant K \sum_{\nu=1}^{\infty} \mu_\nu \mid \Delta \varepsilon_\nu \mid \quad \text{(by Lemma 2)}$$

$$\leqslant K, \qquad \text{by hypothesis.}$$

Lastly,

$$\sum_n \frac{p_n}{P_n P_{n-1}} \sum\nolimits_{13} = \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} p_{n-\nu} \mid \varepsilon_{\nu+1} \mid \mu_\nu$$

$$= \sum_{\nu=1}^{\infty} \mu_\nu \mid \varepsilon_{\nu+1} \mid \sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} p_{n-\nu}$$

$$\leqslant K \sum_{\nu=1}^{\infty} \nu^{-1} \mu_\nu \mid \varepsilon_{\nu+1} \mid \quad \text{(by Lemma 1)}$$

$$\leqslant K, \qquad \text{by hypothesis.}$$

Proof of (2.3.2).

$$\sum\nolimits_2 = \sum_{\nu=1}^{n} \left| \Delta_\nu \left\{ (p_{n-\nu} - p_n) \frac{\varepsilon_\nu}{\nu} \right\} \right| \nu \mu_\nu$$

$$= \sum_{\nu=1}^{n} (p_{n-\nu} - p_n) \frac{|\varepsilon_\nu|}{\nu} \mu_\nu + \sum_{\nu=1}^{n} (p_{n-\nu} - p_n) \mid \Delta \varepsilon_\nu \mid \mu_\nu + \sum_{\nu=1}^{n} |\Delta_n p_{n-\nu-1}| \mid \varepsilon_{\nu+1} \mid \mu_\nu$$

$$= \sum\nolimits_{21} + \sum\nolimits_{22} + \sum\nolimits_{23}, \qquad \text{say.}$$

Now,

$$\sum_n \frac{1}{P_{n-1}} \sum\nolimits_{21} = \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} (p_{n-\nu} - p_n) \frac{|\varepsilon_\nu|}{\nu} \mu_\nu$$

$$= \sum_{\nu=1}^{\infty} \nu^{-1} \mu_\nu \mid \varepsilon_\nu \mid \sum_{n=\nu}^{\infty} \frac{p_{n-\nu} - p_n}{P_{n-1}}$$

$$\leqslant K \sum_{\nu=1}^{\infty} \nu^{-1} \mu_\nu \mid \varepsilon_\nu \mid \quad \text{(by Lemma 4)}$$

$$\leqslant K, \quad \text{by hypothesis.}$$

Next,

$$\sum_n \frac{1}{P_{n-1}} \Sigma_{22} = \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} (p_{n-\nu} - p_n) \mid \Delta \varepsilon_\nu \mid \mu_\nu$$

$$= \sum_{\nu=1}^{\infty} \mu_\nu \mid \Delta \varepsilon_\nu \mid \sum_{n=\nu}^{\infty} \frac{p_{n-\nu} - p_n}{P_{n-1}}$$

$$\leqslant K \sum_{\nu=1}^{\infty} \mu_\nu \mid \Delta \varepsilon_\nu \mid \quad \text{(by Lemma 4)}$$

$$\leqslant K, \quad \text{by hypothesis.}$$

Finally,

$$\sum_n \frac{1}{P_{n-1}} \Sigma_{23} = \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} \mid \Delta_n \, p_{n-\nu-1} \mid \mid \varepsilon_{\nu+1} \mid \mu_\nu$$

$$= \sum_{\nu=1}^{\infty} \mu_\nu \mid \varepsilon_{\nu+1} \mid \sum_{n=\nu}^{\infty} \frac{\mid \Delta_n \, p_{n-\nu-1} \mid}{P_{n-1}}$$

$$\leqslant K \sum_{\nu=1}^{\infty} \frac{\mu_\nu}{P_\nu} \mid \varepsilon_{\nu+1} \mid + K \sum_{\nu=1}^{\infty} \nu^{-1} \mu_\nu \mid \varepsilon_{\nu+1} \mid \quad \text{(by Lemma 3)}$$

$$\leqslant K, \quad \text{by hypothesis.}$$

This terminates the proof of Theorem 1.

2.4. – Proof of Theorems 2 and 3.

We obtain Theorem 2 from Theorem 1 with $\mu_n \equiv 1$, by an appeal to Lemma 5; and we obtain Theorem 3 from Theorem 1, with $\mu_n = \log n$, by an appeal to Lemma 6.

2.5. – We deduce the following corollaries from our theorems (Theorems 1, 2 and 3).

Corollary I. *Let $p_0 > 0$, and let p_n be non-negative and non-increasing. If $t_n^1 = O(1)$, as $n \to \infty$, and if the sequence $\{\lambda_n\}$ is such that*

$$\text{(i)} \quad \sum n^{-1} |\lambda_n| < \infty, \quad \textit{and} \quad \text{(ii)} \quad \sum |\Delta\lambda_n| < \infty,$$

then the series $\sum(P_n/n)\lambda_n a_n$ is summable $|\mathrm{N}, p_n|$.

Corollary II. *Let $p_0 > 0$, and let p_n be non-negative and non-increasing. If $\Phi_1(t) \in \mathrm{BV}(0, \pi)$, and if the sequence $\{\lambda_n\}$ is such that*

$$\text{(i)} \quad \sum n^{-1} |\lambda_n| < \infty. \quad \textit{and} \quad \text{(ii)} \quad \sum |\Delta\lambda_n| < \infty,$$

then the series $\sum (P_n/n)\lambda_n A_n(t)$, at $t = x$, is summable $|\mathrm{N}, p_n|$.

Corollary III. *Let $p_0 > 0$, and let p_n be non-negative and non-increasing. If*

$$\int_0^t u\, |d\Phi_1(u)| = O(t), \qquad 0 \leqslant t \leqslant \pi,$$

and if the sequence $\{\lambda_n\}$ is such that

$$\text{(i)} \quad \sum n^{-1} |\lambda_n| < \infty, \quad \textit{and} \quad \text{(ii)} \quad \sum |\Delta\lambda_n| < \infty,$$

then the series $\sum \big(P_n/(n \log n)\big)\lambda_n A_n(t)$, at $t = x$, is summable $|\mathrm{N}, p_n|$.

We remark that, if in these corollaries we take $\{\lambda_n\}$ to be a non-increasing sequence such that the series $\sum n^{-1}\lambda_n$ is convergent, then, by Lemma 7, the conditions on the sequence $\{\lambda_n\}$ are automatically satisfied and hence these corollaries are more general than the Theorems C and D.

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