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On the Absolute Nörlund Summability of a Fourier Series. I. (**)

1. - Definitions and notations.

1.1. - Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \dots + p_n , \qquad P_{-1} = p_{-1} = 0 .$$

The sequence-to-sequence transformation:

$$t_n = \sum_{\nu=0}^{n} p_{n-\nu}\, s_\nu / P_n \qquad (P_n \neq 0) \tag{1.1.1}$$

defines the sequence $\{t_n\}$ of NÖRLUND means (1) of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable (N, p_n) to the sum s if $\lim_{n\to\infty} t_n$ exists and is equal to s, and is said to be absolutely summable (N, p_n), or summable $|N, p_n|$ (2), if the sequence $\{t_n\}$ is of bounded variation (3), that is, $\sum_n |t_n - t_{n-1}| \leqslant K$ (4).

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(1) NÖRLUND [4]. See also WORONOI [10].

(2) MEARS [3].

(3) Symbolically, $\{t_n\} \in BV$; similarly by $F(t) \in BV\,(h, k)$ we shall mean that $F(t)$ is a function of bounded variation over the interval (h, k).

(4) Throughout this paper K denotes a positive constant, not necessarily the same at each occurrence.

1.2. – Let $f(t)$ be a periodic function with period 2π and integrable in the LEBESGUE sense over $(-\pi, \pi)$. We assume, without any loss of generality, that the constant term in the FOURIER series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)\, dt = 0 \,, \tag{1.2.1}$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) \,. \tag{1.2.2}$$

We write throughout

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}, \qquad \lambda_{n,k}(t) = \frac{\sin (n-k)t}{n-k} \,,$$

$$R_n = (n+1)\, p_n/P_n \,, \qquad S_n = \frac{1}{P_n} \sum_{\nu=0}^{n} \frac{P_\nu}{\nu+1} \,, \qquad c_n = \sum_{k=n}^{\infty} \frac{1}{(k+2)P_k} \,,$$

$\tau = [\pi/t]$, that is, the greatest integer not greater than π/t; for any sequence $\{\sigma_n\}$, $\Delta\sigma_n = \sigma_n - \sigma_{n+1}$.

2. - Introduction.

2.1. – Concerning the summability $|N, p_n|$ of the FOURIER series at a point, the following was proved by PATI.

Theorem A (5). If $\varphi(t) \in BV(0, \pi)$ and $\{p_n\}$ is a positive, monotonic sequence such that $P_n \to \infty$, as $n \to \infty$, and $\{R_n\} \in BV$ and $\{S_n\} \in BV$, then the FOURIER series of $f(t)$, at $t = x$, is summable $|N, p_n|$.

Recently, VARSHNEY proved the following.

Theorem B (6). If $\varphi(t) \in BV(0, \pi)$, and $\{p_n\}$ is a positive sequence

(5) PATI [5], [6]. This theorem contains as a special case a well-known previous result of BOSANQUET, on the absolute CESÀRO summability of a FOURIER series. See BOSANQUET [1].

(6) VARSHNEY [9].

such that $P_n \to \infty$, as $n \to \infty, \{R_n\} \in \mathrm{BV}$ and

$$P_n c_n \leqslant K \qquad (n = 0, 1, 2, \ldots), \tag{2.1.1}$$

then the FOURIER series of $f(t)$, at $t = x$, is summable $|\mathrm{N}, p_n|$.

It has been very recently shown by PATI that in Theorem A the condition of monotonicity of $\{p_n\}$ and « $P_n \to \infty$, as $n \to \infty$ » can be easily avoided, that in Theorem B « $P_n \to \infty$, as $n \to \infty$ » is implied by condition (2.1.1), and that the sets of conditions:

« $\{R_n\} \in \mathrm{BV}$ and $\{S_n\} \in \mathrm{BV}$ » and « $\{R_n\} \in \mathrm{BV}$ and (2.1.1) » are equivalent (7).

It may be remarked that the case: « $\{p_n\}$ is monotonic non-increasing » is the most important in view of the fact that it covers the case $p_n = A_n^{\delta-1}$, $0 < \delta < 1$, giving the result of $|C, \delta|$ summability for the FOURIER series of $f(t)$ at $t = x$.

We give here a very brief proof of the Theorem B for this case, i.e., when $\{p_n\}$ is monotonic non-increasing.

2.2. – We require the following lemmas.

Lemma 1 (8). *Uniformly for* $0 < t \leqslant \pi$,

$$\left| \sum_{\nu=m}^{n} \frac{\sin \nu t}{\nu} \right| \leqslant K ,$$

where m *and* n *are any positive integers such that* $n \geqslant m$.

Lemma 2 (9). *If* $\{q_n\}$ *is non-negative and non-increasing, then, for* $0 \leqslant a \leqslant b \leqslant \infty$, $0 \leqslant t \leqslant \pi$, *and any* n:

$$\left| \sum_{k=a}^{b} q_k \, e^{i(n-k)t} \right| \leqslant K \, Q_\tau ,$$

where $\tau = [\pi/t]$, *and* $Q_n = q_0 + q_1 + \ldots + q_n$.

(7) See PATI [7].

(8) TITCHMARSH [8], § 1. 76.

(9) MCFADDEN [2], Lemma 5.11, p. 182; this is originally due to HILLE and TAMARKIN.

3. - Proof of Theorem B for monotonic non-increasing $\{p_n\}$ (10).

As in PATI [5], it is enough to show that, uniformly for $0 < t \leqslant \pi$,

$$\sum \equiv \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} (P_n p_k - p_n P_k)\lambda_{n,k}(t) \right| \leqslant K .$$

Now:

$$\sum \leqslant \sum_n \frac{1}{(n+1) P_n P_{n-1}} \left| \sum_{k=0}^{n-1} \left\{P_n p_k (k+1) - p_n P_k (n+1)\right\} \lambda_{n,k}(t) \right|$$

$$+ \sum_n \frac{1}{(n+1) P_n P_{n-1}} \left| \sum_{k=0}^{n-1} \left\{P_n p_k (n+1) - P_n p_k (k+1)\right\} \lambda_{n,k}(t) \right|$$

$$= \sum\nolimits_1 + \sum\nolimits_2 , \quad \text{say.}$$

We have:

$$\sum\nolimits_2 = \sum_n \frac{1}{(n+1) P_{n-1}} \left| \sum_{k=0}^{n-1} p_k \sin (n-k)t \right|$$

$$\leqslant \sum_{n=1}^{\tau} \frac{1}{(n+1) P_{n-1}} \left| \sum_{k=0}^{n-1} p_k \sin (n-k)t \right|$$

$$+ \sum_{n=\tau+1}^{\infty} \frac{1}{(n+1) P_{n-1}} \left| \sum_{k=0}^{n-1} p_k \sin (n-k)t \right|$$

$$= \sum\nolimits_{21} + \sum\nolimits_{22} , \quad \text{say.}$$

Also, since

$$|\sin (n-k)t| \leqslant (n-k)t \leqslant nt ,$$

we have

$$\sum\nolimits_{21} \leqslant t \sum_{n=1}^{\tau} \frac{n}{(n+1) P_{n-1}} \sum_{k=0}^{n-1} p_k$$

$$\leqslant Kt\tau$$

$$\leqslant K .$$

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(10) That the hypothesis: (2.1.1.) of Theorem B is equivalent to $\{S_n\} \in B$, i.e., $\{S_n\}$ is a bounded sequence, whenever $\{p_n\}$ is a positive sequence and $\{R_n\} \in BV$, has been pointed out recently in: H. P. DIKSHIT, *Absolute summability of a Fourier series by Nörlund means*, forthcoming in Math. Z.

By virtue of Lemma 2,

$$\sum_{22} \leqslant K\, P_\tau\, c_\tau \leqslant K\,,$$

by hypothesis (2.1.1).

Lastly,

$$\sum_1 = \sum_{n=1}^{\infty} \frac{1}{(n+1)\,P_{n-1}} \,\Big|\, \sum_{k=0}^{n-1} P_k\,(R_k - R_n)\,\lambda_{n,k}(t)\,\Big|$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)\,P_{n-1}} \,\Big|\, \sum_{k=0}^{n-1} P_k\,\lambda_{n,k}(t) \sum_{\nu=k}^{n-1} \Delta R_\nu \,\Big|$$

$$\leqslant \sum_{n=1}^{\infty} \frac{1}{(n+1)\,P_{n-1}} \sum_{\nu=0}^{n-1} |\,\Delta R_\nu\,|\,\Big|\, \sum_{k=0}^{\nu} P_k\,\lambda_{n,k}(t)\,\Big|$$

$$\leqslant K \sum_{n=1}^{\infty} \frac{1}{(n+1)\,P_{n-1}} \sum_{\nu=0}^{n-1} |\,\Delta R_\nu\,|\, P_\nu\,,$$

by ABEL's Lemma and Lemma 1,

$$= K \sum_{\nu=0}^{\infty} |\,\Delta R_\nu\,|\, P_\nu\, c_\nu$$

$$\leqslant K \sum_{\nu=0}^{\infty} |\,\Delta R_\nu\,|\,, \qquad \text{by (2.1.1)},$$

$$\leqslant K\,,$$

by the hypothesis $\{R_n\} \in BV$.

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