Riv. Mat. Univ. Parma (2) 7 (1966), 185-191

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A Representation Theorem for «Regular» Hemimorphisms Between Boolean Algebras. (**)

Introduction.

In [3] I defined a regular function to be a normal and completely additive (see JÓNNSON and TARSKI: [2]) function f from a complete Boolean Algebra $\mathscr{A}$ to a complete Boolean Algebra $\mathscr{A}'$ such that $fx = 0 \iff x = 0$. The concept of regular function was meant to generalize that of point mapping between two sets, in a sense made precise in what follows. For the purposes of [3] it was convenient to bound oneself to complete Boolean Algebras and complete hemimorphisms, but I will presently abandon these restrictions, thus referring to the more general concept of *regular hemimorphism*.

Let S, S' be two (non void) sets, $F: S \to S'$. Consider the map F^* defined by: $F^*X = F(X) = \{Fx: x \in X\}$. We have that:

(1) $$F^*X = \varnothing \iff X = \varnothing;$$

(2) $$F^*(X \cup Y) = F^*X \cup F^*Y, \qquad (X,\ Y \subset S).$$

If we indicate by $\mathscr{B}(M)$ the complete field of subsets of a set M, then (1) and (2) show that $F^*: \mathscr{B}(S) \to \mathscr{B}(S')$ is a special kind of hemimorphism between Boolean Algebras. In general, given two Boolean Algebras, $\mathscr{A}$, $\mathscr{A}'$, a function

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(**) Ricevuto il 6-V-1966. Lavoro eseguito nell'ambito dell'attività dei Gruppi di ricerca matematici del C. N. R. (Gruppo n. 37, anno 1965-66).

$f: \mathscr{A} \to \mathscr{A}'$ will be said to be a *regular hemimorphism* if

$$(1') \qquad f x = 0 \iff x = 0$$

and

$$(2') \qquad f(x \vee y) = f x \vee f y, \qquad (x, y \in \mathscr{A}).$$

Thus, (1), (2) prove that F^* is a regular hemimorphism: it will be said to be *induced* by the point function F. The natural question arises whether any regular hemimorphism is induced (up to isomorphisms) by a point mapping. The answer is affirmative: namely I will prove that there are representations (¹) as fields of sets for $\mathscr{A}$, $\mathscr{A}'$, say (φ, S), (φ', S') respectively, and there is a function $F: S \to S'$, such that $F(\varphi x) = \varphi' f x$, $\forall x \in \mathscr{A}$.

The method used for this proof is, apart from a final trick, that of immerging conveniently the two algebras into complete fields of sets $\mathscr{B}$, $\mathscr{B}'$, then extending f to a function from $\mathscr{B}$ to $\mathscr{B}'$. The path is now traced and most results come, practically « gratis », from those papers which deal with the extending of an additive operator in a Boolean Algebra (see, e.g., JÓNNSON and TARSKI [2], SERVI [4]).

The proofs given in section 2 are very similar to those contained in my paper [4], yet I prefer to repeat them (adjusted to the new more general situation (²) and with some minor changes) for the sake of clarity and completeness.

1. - Notations.

If M is any set, $\mathscr{B}(M)$ will indicate the field of all the subsets of M. If S, S' are two sets, $F: S \to S'$ and $G: S \to \mathscr{B}(S')$, then F^*, G^* will be the two functions from $\mathscr{B}(S)$ to $\mathscr{B}(S')$ defined respectively by:

$$\left.\begin{array}{ll} (3) \qquad & F^* X = \{ Fx : x \in X \} \\ (4) \qquad & G^* X = \bigcup_{x \in X} Gx \end{array}\right\} \qquad (X \in \mathscr{B}(S)).$$

(¹) A pair (φ, S) is said to be a *representation* for a Boolean Algebra $\mathscr{A}$ if S is a set and φ is a monomorphism of $\mathscr{A}$ into $\mathscr{B}(S)$, the complete field of subsets of S.

(²) In [4] the functions we dealt with were « endofunctions » (i. e. from an algebra $\mathscr{A}$ to the same algebra) and furthermore they were subjected to additional conditions, as that of being completely additive, so that here we are facing a twofold generalization.

Observe that F^* and G^* preserve (infinite) unions.

The restriction of a function f to a subset S of its domain will be indicated as usual by $f \mid S$.

2. – In this section $\mathscr{A}$ will be a field of subsets of a set S, $\mathscr{A}'$ will be a field of subsets of a set S', $f\colon \mathscr{A} \to \mathscr{A}'$ will be a hemimorphism and $F\colon S \to \mathscr{B}(S')$ will be defined by

$$Fx = \bigcap_{\substack{X \in \mathscr{A} \\ x \in X}} fX \qquad (x \in S).$$

We can now state the following

Theorem 1. *$F^* \mid \mathscr{A} = f$ if and only if for every $\mathscr{F} \subset \mathscr{A}$*

$$\bigcup_{X \in \mathscr{F}} X \in \mathscr{A} \implies f\Big(\bigcup_{X \in \mathscr{F}} X\Big) = \bigcup_{X \in \mathscr{F}} fX \,. \tag{5}$$

Proof.

If $F^* \mid \mathscr{A} = f$ holds, then (5) is an immediate consequence of the fact that F^* preserves unions (see above section).

Viceversa, observe first that $F^*X \subset fX$, for every $X \in \mathscr{A}$: the proof for this is quite obvious and in any case it is formally identical with that given in [4], thm. 3 (i). Let's now prove the converse inclusion $fX \subset F^*X$ $(X \in \mathscr{A})$. By contradiction, suppose there is an $x \in fX$ such that $x \notin F^*X$. By definition of F^* we get then:

$$\forall\, y \in X, \quad \exists Y \in \mathscr{A} \quad \text{such that } y \in Y \text{ and } x \notin fY. \tag{6}$$

Put $\mathscr{F} = \{ Z \in \mathscr{A} \colon Z \subset X \text{ and } x \notin f\,Z \}$. Then, by (6), we get $\bigcup_{Z \in \mathscr{F}} Z \supset X$. The converse inclusion follows from definition of $\mathscr{F}$, therefore

$$\bigcup_{Z \in \mathscr{F}} Z = X \,. \tag{7}$$

Since $X \in \mathscr{A}$, by our assumption (5) this implies $f(\bigcup_{Z \in \mathscr{F}} Z) = \bigcup_{Z \in \mathscr{F}} f\,Z$, i.e. $f\,X = \bigcup_{Z \in \mathscr{F}} f\,Z$, and this is a contradiction, since $x \in fX$, but $x \notin fZ$, $\forall Z \in \mathscr{F}$.

Remark.

If, roughly speaking, S is the dual space of $\mathscr{A}$, then we know that S is compact (in the topology having $\mathscr{A}$ as a basis). There follows that condition (5) is always satisfied. To prove this, suppose $\bigcup_{x\in\mathscr{F}} X \in \mathscr{A}$. Then $\bigcup_{x\in\mathscr{F}} X$ is a clopen subset of a compact space, hence is compact: therefore the open cover $\mathscr{F}$ has a finite subcover, say $\{X_1, \ldots, X_n\} \subset \mathscr{F}$, $\bigcup_{x\in\mathscr{F}} X = X_1 \cup \ldots \cup X_n$. There follows: $f(\bigcup_{x\in\mathscr{F}} X) = f(X_1 \cup \ldots \cup X_n) = fX_1 \cup \ldots \cup fX_n \subset \bigcup_{x\in\mathscr{F}} fX$. On the other hand, $f(\bigcup_{x\in\mathscr{F}} X) \supset \bigcup_{x\in\mathscr{F}} fX$, since f is a hemimorphism, thus $f(\bigcup_{x\in\mathscr{F}} X) = \bigcup_{x\in\mathscr{F}} fX$.

3. – Let $\mathscr{A}$, $\mathscr{A}'$ be any two Boolean algebras, $f: \mathscr{A} \to \mathscr{A}'$ any hemimorphism, (φ', S') any representation for $\mathscr{A}'$, (φ, S) the STONE representation for $\mathscr{A}$. As a corollary of the preceding section, we then have that

Lemma 1.

There exists a function $F: S \to \mathscr{B}(S')$ *such that*

$$F^*\varphi = \varphi' f \,. \tag{8}$$

This is not yet the final goal, though, becouse F is not from S to S'. We now undertake an intermediate step for which regularity is needed.

Let $f: \mathscr{A} \to \mathscr{A}'$ be a regular hemimorphism and let S, S', φ, φ' be as before. Consider the set

$$N = \{x \in S:\ Fx = \varnothing\}.$$

We claim:

Lemma 2.

(i) $S_0 = S \sim N$ *is a representative set for* $\mathscr{A}$, *i.e. if* ϱ *indicates the trace homomorphism,*

$$\varrho X = X \sim N \qquad (X \subset S)\,,$$

then ϱ *is an isomorphism of* $\varphi(\mathscr{A})$ *onto* $\varrho(\varphi(\mathscr{A}))$;

(ii) *if we define* $F_0: S_0 \to \mathscr{B}(S')$ *to be the restriction (to* S_0*) of* F, *then the following holds:*

$$F_0^* \varrho\varphi = \varphi' f \,.$$

Proof.

For (i) it is enough to show that $X \sim N = \varnothing \Longrightarrow X = \varnothing$, $\forall X \in \varphi(\mathscr{A})$. Let $X \sim N = \varnothing$, with $X \in \varphi(\mathscr{A})$; then $X \subset N$; since F^* preservers unions, it is an increasing function and hence $F^*X \subset F^*N = \bigcup_{x \in N} Fx = \varnothing$. But $X \in \varphi(\mathscr{A})$, therefore $X = \varphi a$, for a suitable $a \in \mathscr{A}$, and hence, by (8), $\varphi' f a = F^* \varphi a = F^* X = \varnothing$. Since f is regular and φ, φ' are monomorphisms, there follows $X = \varnothing$.

As for (ii) we have:

$$F_0^* \varrho \varphi a = \bigcup_{x \in \varrho\varphi a} Fx \subset \bigcup_{x \in \varphi a} Fx = F^* \varphi a = \varphi' f a.$$

The converse inclusion is proved as follows: let $y \in \varphi' f a$; then

$$y \in F^* \varphi a = \bigcup_{x \in \varphi a} Fx;$$

$\exists x \in \varphi a$, with $y \in Fx$.

Hence $Fx \neq \varnothing$ and thus $x \notin N$, whence $x \in \varrho\varphi a$. There follows

$$y \in \bigcup_{x \in \varrho\varphi a} Fx\,,$$

and finally $\varphi' f a \subset F_0^* \varrho \varphi a$.

Remark.

Of course, becouse F_0 is the restriction to $S \sim N$ of F, we never have $F_0 x = \varnothing$, $x \in S_0$.

4. – Putting together the results of the two preceding sections, we have:

Lemma 3.

Let $f\colon \mathscr{A} \to \mathscr{A}'$ be any regular hemimorphism and let (φ', S') be any representation for $\mathscr{A}'$. Then there exists a suitable representation, say (φ_0, S_0), for $\mathscr{A}$ and a function $F_0\colon S_0 \to \mathscr{B}(S')$ such that

(9) $$F_0^* \varphi_0 = \varphi' f$$

and

(10) $$F_0\, x \neq \varnothing \qquad (x \in S_0)\,.$$

In order to get a point (single valued) function F, we again change representation for $\mathscr{A}$, but for the last time.

For each $x \in S_0$, put

$$S_x = \{x\} \times F_0\, x\ (^3).$$

Let π indicate second projection:

$$\pi(x,\ y) = y \qquad ((x,\ y) \in S_0 \times S')\ .$$

Put $S = \bigcup_{x \in S_0} S_x$ and define φ: $\mathscr{A} \to \mathscr{B}(S)$ as follows:

$$\varphi a = \bigcup_{x \in \varphi_0 a} S_x \qquad (a \in \mathscr{A})\ . \tag{11}$$

My claim is that φ *is a monomorphism and therefore* $(\varphi,\ S)$ *is a representation for* $\mathscr{A}$.

By straightforward computation one checks that φ preserves Boolean unions, 0, 1; to prove it preserves intersections, one has to use the fact that if $x \neq y$, then $S_x \cap S_y = \varnothing$ $(x,\ y \in S_0)$. Finally, suppose $\varphi a = \varnothing$. Since $F_0\, x \neq \varnothing$ (lemma 3, formula (10)), also $S_x \neq \varnothing$; by (11) there follows $\varphi_0 a = \varnothing$ and hence $a = 0$, since φ_0 is a monomorphism. This proves φ is a monomorphism too.

Now, we have to define F: $S \to S'$. Since $S \subset S_0 \times S'$, it makes sense to define $F = \pi \mid S$. If we can prove

$$\pi^* \varphi = F_0^* \varphi_0,$$

then our aim will have been reached, thanks to lemma 3, formula (9).

We have:

Let $y \in F_0^* \varphi_0 a = \bigcup_{x \in \varphi_0 a} F_0 x$; then there exists an $x \in \varphi_0 a$ such that $y \in F_0 x$. Thus $(x,\ y) \in S_x \subset \bigcup_{z \in \varphi_0 a} S_z = \varphi a$. From this we get: $\pi(x,\ y) \in \pi(\varphi a)$, i.e. $y \in \pi^* \varphi a$. Thus we proved $F_0 \varphi_0 a \subset \pi^* \varphi a$.

Conversely, let y be an element of $\pi^* \varphi a$, i.e. of $\pi(\varphi a)$. Then

$$y = \pi t, \tag{12}$$

for a suitable $t \in \varphi a$. By the definition of φ, there exists an $x \in \varphi_0 a$ with $t \in S_x$ and by the definition of S_x, there is a $z \in F_0\, x$ such that $t = (x,\ z)$. By (12), $y = z$, hence $y \in F_0\, x$; from this and from $x \in \varphi_0\, a$ we get $y \in \bigcup_{u \in \varphi_0 a} F_0 u = F_0^* \varphi_0\, a$ and equality is then proved.

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(3) Clearly S_x is a subset of the Cartesian product $S_0 \times S'$.

Taking into account formula (9), we can finally state our main theorem.

Theorem 2. *Given any two Boolean Algebras $\mathscr{A}$, $\mathscr{A}'$, any regular hemimorphism $f: \mathscr{A} \to \mathscr{A}'$ and given any representation (φ', S') for $\mathscr{A}'$, there exists a suitable representation (φ, S) for $\mathscr{A}$ and a function $F: S \to S'$ such that*

$$F^* \varphi = \varphi' f .$$

Remark.

If we give up the requirement that $\mathscr{A}'$ be arbitrarily represented, then a simplification follows in section 3. Let (φ', S') be the STONE representation for $\mathscr{A}'$; then the set N is empty, and we need not change representation for $\mathscr{A}$. The proof for this follows from compactness of dual space. Using notations of section 2, we have to prove that $Fx \neq \varnothing$, $\forall x \in S$. If it were $Fx = \varnothing$, by compactness there would exist a finite number of elements of $\mathscr{A}$, say $X_1, \ldots, X_n$, such that $x \in X_1 \cap \ldots \cap X_n$ and $fX_1 \cap \ldots \cap fX_n = \varnothing$. This implies $f(X_1 \cap \ldots \cap X_n) = \varnothing$, since f is an increasing function and, by regularity of f, $X_1 \cap \ldots \cap X_n = \varnothing$, a contradiction.

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Summary.

A hemimorphism f between two Boolean algebras $\mathscr{A}, \mathscr{A}'$ is said to be regular if for every $x \in \mathscr{A}$, $fx = 0$ is equivalent to $x = 0$. We prove that for any regular hemimorphism $f: \mathscr{A} \to \mathscr{A}'$ it is possible to find two sets S, S' and a function $F: S \to S'$ such that $\mathscr{A}, \mathscr{A}'$ can be represented as fields of subsets of S, S' respectively and the transform of f in these representations is induced by F.

Sunto.

Un emimorfismo f fra due algebre di Boole $\mathscr{A}, \mathscr{A}'$ si dice regolare se $fx = 0$ equivale ad $x = 0$, per ogni $x \in \mathscr{A}$. Nel presente lavoro si dimostra che per ogni emimorfismo regolare $f: \mathscr{A} \to \mathscr{A}'$ si possono trovare due insiemi S, S' ed una funzione $F: S \to S'$ tali che $\mathscr{A}, \mathscr{A}'$ siano rappresentabili come campi di sottoinsiemi di S, S' rispettivamente, e che la trasformata di f in tali rappresentazioni sia indotta da F.

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