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Some Uses of the Basic Properties of Meijer Transform to Integral Function of Two Complex Variables. (**)

1. – Let

$$F(z_1, z_2) = \sum_{n_1, n_2 = 0}^{\infty} \frac{a_{n_1, n_2}}{n_1!\, n_2!} z_1^{n_1} z_2^{n_2}$$

be an integral function of two complex variables z_1 and z_2. Denote

$$M_F(r) = \max_{|z_1| + |z_2| = r} |F(z_1, z_2)|$$

the maximum modulus of $F(z_1, z_2)$.

M. M. DŽRBAŠYAN [1] has given the following definition of order:

The integral function $F(z_1, z_2)$ is said to be of finite order ϱ, if

$$\limsup_{r \to \infty} \frac{\log \log M_F(r)}{\log r} = \varrho \qquad (0 \leqslant \varrho < \infty). \tag{1.1}$$

MEIJER [2] introduced the integral transformation

$$\varphi(p) = p \int_0^{\infty} e^{-\frac{1}{2}pt} W_{k+\frac{1}{2}, m}(pt)\, (pt)^{-k-\frac{1}{2}} f(t)\, dt, \qquad \operatorname{Re} p > 0, \tag{1.2}$$

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where $W_{k,m}(z)$ is the WHITTAKER's confluent hypergeometric function and (1.2) is symbolically denoted by

$$\varphi(p) = M[f(t);\ k+\tfrac{1}{2}, m],$$

which is a generalisation of the integral equation

$$(1.3)\qquad \varphi(p) = p\int_0^\infty e^{-pt} f(t)\,\mathrm{d}t, \qquad \operatorname{Re} p > 0,$$

the well known LAPLACE transform and is symbolically denoted by

$$\varphi(p) = L[f(t)].$$

The object of this paper is to evaluate some new type of integrals involving integral functions of two complex variables of order ϱ, based on the properties of MEIJER transform of one variable and the Eulerian integral of the first kind. The particular case when the integral function is of order one has also been studied.

2. – Theorem 1. *Let* $|\zeta_j| \neq 0$, $|\arg \zeta_j| < \pi/(2\varrho)$, $j = 1, 2$, *and let*

$$F(z_1, z_2) = \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!\,n_2!} z_1^{n_1} z_2^{n_2},$$

be an integral function of two complex variables z_1 *and* z_2 *of order* $\varrho (0 < \varrho < \infty)$; *then for* $\arg\zeta_1 = \arg\zeta_2$, *we have*

$$(2.1)\quad \begin{cases} J_{n_1,n_2}(\zeta_1, \zeta_2) = \\ = \int_0^\infty\int_0^\infty e^{-\frac{1}{2}(t_1\zeta_1+t_2\zeta_2)^\varrho}(t_1\zeta_1+t_2\zeta_2)^{\nu\varrho-2}\, W_{k+\frac{1}{2},m}\left((t_1\zeta_1+t_2\zeta_2)^\varrho\right)F(t_1, t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ = \dfrac{1}{\varrho}\displaystyle\sum_{n_1,n_2=0}^{\infty} \frac{b_{n_1,n_2}}{(n_1+n_2+1)!}\,\zeta_1^{-n_1-1}\zeta_2^{-n_2-1}, \end{cases}$$

where [1]

$$b_{n_1,n_2} = a_{n_1,n_2} \frac{\Gamma_\times\left(\nu + \dfrac{n_1+n_2}{\varrho} + \frac{1}{2} \pm m\right)}{\Gamma\left(\nu + \dfrac{n_1+n_2}{\varrho} + \frac{1}{2} - k\right)},$$

provided $\mathrm{Re}\,(\nu + \frac{1}{2} \pm m) > 0$.

Proof.

Let us first take $\mathrm{Re}\, p > 0$ and consider the integral

$$I_{n_1,n_2}(p) = \tag{2.2}$$

$$= \int_0^\infty \int_0^\infty e^{-\frac{1}{2}p(x_1+x_2)^\varrho} (x_1 + x_2)^{\nu\varrho-2} W_{k+\frac{1}{2},m}\big(p(x_1+x_2)^\varrho\big)\, x_1^{n_1} x_2^{n_2}\, \mathrm{d}x_1\, \mathrm{d}x_2\,,$$

where n_1 and n_2 are positive integers. Changing the variables $x_1 = t(1-u)$, $x_2 = tu$, $0 \leqslant u \leqslant 1$, $0 \leqslant t < +\infty$, we have

$$I_{n_1,n_2}(p) =$$

$$= \int_0^\infty \int_0^1 e^{-\frac{1}{2}pt^\varrho} t^{n_1+n_2+\nu\varrho-2}\, W_{k+\frac{1}{2},m}(pt^\varrho)\,(1-u)^{n_1} u^{n_2} \frac{\partial(x_1, x_2)}{\partial(t, u)}\, \mathrm{d}t\, \mathrm{d}u$$

$$= \int_0^\infty \int_0^1 e^{-\frac{1}{2}pt^\varrho} t^{n_1+n_2+\nu\varrho-1} W_{k+\frac{1}{2},m}(pt^\varrho)\,(1-u)^{n_1} u^{n_2}\, \mathrm{d}t\, \mathrm{d}u.$$

Evaluating u-integral with the help of the Eulerian integral of the first kind (cfr. [3], p. 212), we obtain

$$I_{n_1,n_2}(p) = \frac{n_1!\, n_2!}{(n_1+n_2+1)!} \int_0^\infty e^{-\frac{1}{2}pt^\varrho}\, t^{n_1+n_2+\nu\varrho-1}\, W_{k+\frac{1}{2},m}(pt^\varrho)\, \mathrm{d}t$$

$$= \frac{n_1!\, n_2!}{\varrho(n_1+n_2+1)!}\, p^{-\nu-\frac{n_1+n_2}{\varrho}} \int_0^\infty e^{-\frac{1}{2}x} x^{\nu+\frac{n_1+n_2}{\varrho}-1} W_{k+\frac{1}{2},m}(x)\, \mathrm{d}x\,.$$

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[1] The symbol $\Gamma_\times(u \pm v)$ denotes $\Gamma(u+v)\cdot\Gamma(u-v)$.

Now evaluating x-integral with the help of GOLDSTEIN's integral (cfr. [4], p. 114) ([2])

$$\int_0^\infty x^{l-1} e^{-(\eta^2+1/2)x} W_{k,m}(x)\, dx = \frac{\Gamma_\times(l+\frac{1}{2}\pm m)}{\Gamma(l+1-k)}\, {}_2F_1 \left[\begin{matrix} l+\frac{1}{2}\pm m \\ l+1-k \end{matrix} ; -\eta^2 \right],$$

$$\operatorname{Re}(l+\tfrac{1}{2}\pm m) > 0\,, \qquad \operatorname{Re}(\eta^2+1) > 0\,,$$

leads to

$$(2.3) \qquad I_{n_1,n_2}(p) = \frac{n_1!\; n_2!}{\varrho(n_1+n_2+1)!}\; \frac{\Gamma_\times\left(\nu + \dfrac{n_1+n_2}{\varrho} + \frac{1}{2} \pm m\right)}{\Gamma\left(\nu + \dfrac{n_1+n_2}{\varrho} + \frac{1}{2} - k\right)}\; p^{-\nu-\frac{n_1+n_2}{\varrho}},$$

$$\operatorname{Re}(\nu+\tfrac{1}{2}\pm m) > 0\,, \qquad \operatorname{Re} p > 0\,.$$

Let $\arg \zeta_1 = \arg \zeta_2 = \alpha$ and if we denote $p = e^{i\alpha\varrho}$, where $\operatorname{Re} p = \cos(\alpha\varrho) > 0$, then from (2.1) we obtain

$$(2.4) \quad \left\{ \begin{aligned} & J_{n_1,n_2}(\zeta_1\,,\ \zeta_2) = \\ &= \int_0^\infty\!\!\int_0^\infty e^{-1/2(t_1\zeta_1+t_2\zeta_2)^\varrho} (t_1\zeta_1+t_2\zeta_2)^{\nu\varrho-2}\, W_{k+1/2,m}\left((t_1\zeta_1+t_2\zeta_2)^\varrho\right) \cdot \\ &\qquad \cdot \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!\; n_2!}\, t_1^{n_1} t_2^{n_2}\, dt_1\, dt_2 \\ &= e^{i\alpha(\nu\varrho-2)} \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!\; n_2!} \int_0^\infty\!\!\int_0^\infty e^{-1/2 p(t_1|\zeta_1|+t_2|\zeta_2|)^\varrho} (t_1|\zeta_1|+t_2|\zeta_2|)^{\nu\varrho-2} \cdot \\ &\qquad \cdot W_{k+1/2,m}\left(p(t_1|\zeta_1|+t_2|\zeta_2|)^\varrho\right) t_1^{n_1} t_2^{n_2}\, dt_1\, dt_2\,. \end{aligned} \right.$$

([2]) The symbol ${}_2F_1\left[\begin{matrix} e\pm f \\ g \end{matrix} ; x\right]$ denotes ${}_2F_1\left[\begin{matrix} e+f,\, e-f \\ g \end{matrix} ; x\right]$.

Replacing $t_1 \mid \zeta_1 \mid$ by x_1 and $t_2 \mid \zeta_2 \mid$ by x_2 and evaluating the integrals, we get

$$J_{n_1,n_2}(\zeta_1\,,\ \zeta_2) =$$

$$= e^{i\alpha(\nu\varrho-2)} \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!\ n_2!} \frac{1}{|\zeta_1|^{n_1+1} |\zeta_2|^{n_2+1}} \cdot$$

$$\cdot \int_0^\infty \int_0^\infty e^{-\frac{1}{2}p(x_1+x_2)^\varrho} (x_1 + x_2)^{\nu\varrho-2}\, W_{k+\frac{1}{2},m}\left(p(x_1+x_2)^\varrho\right) x_1^{n_1} x_2^{n_2}\, dx_1\, dx_2$$

$$= \frac{e^{i\alpha(\nu\varrho-2)}}{\varrho} \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{(n_1+n_2+1)!} \frac{\Gamma_{\times}\left(\nu + \dfrac{n_1+n_2}{\varrho} + \frac{1}{2} \pm m\right)}{\Gamma\left(\nu + \dfrac{n_1+n_2}{\varrho} + \frac{1}{2} - k\right)} \frac{p^{-\nu-(n_1+n_2)/\varrho}}{|\zeta_1|^{n_1+1} |\zeta_2|^{n_2+1}},$$

$$\operatorname{Re}(\nu + \tfrac{1}{2} \pm m) > 0\,, \qquad \operatorname{Re} p > 0\,,$$

due to the relation (2.3).

Thus under the conditions $|\zeta_j| \neq 0$, $\arg \zeta_1 = \arg \zeta_2$, $|\arg \zeta_j| < \pi/(2\varrho)$, $j = 1, 2$ and an appeal to analytic continuation, we obtain

$$J_{n_1,n_2}(\zeta_1\,,\ \zeta_2) = \frac{1}{\varrho} \sum_{n_1,n_2=0}^{\infty} \frac{b_{n_1,n_2}}{(n_1+n_2+1)!} \zeta_1^{-n_1-1} \zeta_2^{-n_2-1}\,,$$

where

$$b_{n_1,n_2} = a_{n_1,n_2} \frac{\Gamma_{\times}\left(\nu + \dfrac{n_1+n_2}{\varrho} + \frac{1}{2} \pm m\right)}{\Gamma\left(\nu + \dfrac{n_1+n_2}{\varrho} + \frac{1}{2} - k\right)}\,, \qquad \operatorname{Re}(\nu + \tfrac{1}{2} \pm m) > 0\,,$$

provided the change of order of integration and summation in (2.4) is justified.

Regarding the change of order of integration and summation in (2.4), we note that $F(z_1\,,\ z_2)$ is an integral function of the variables z_1 and z_2 and so the series

$$\sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!\ n_2!} z_1^{n_1} z_2^{n_2}\,,$$

converges uniformly for $|z_j| < R_j$, $j = 1, 2$, for R_j arbitrary large and the integral converges uniformly and absolutely for $\mathrm{Re}\,(\nu + \frac{1}{2} \pm m) > 0$ and $\mathrm{Re}\, p > 0$. Also on using ZEMLJAKOV's theorem (cfr. [1], p. 258), we find that

$$\lim_{n_1+n_2\to\infty} \sup \left| \frac{b_{n_1,n_2}}{(n_1+n_2+1)!} \right|^{1/(n_1+n_2)} = A^{1/\varrho},$$

where

$$b_{n_1,n_2} = a_{n_1,n_2} \frac{\Gamma_\times\left(\nu + \dfrac{n_1+n_2}{\varrho} + \frac{1}{2} \pm m\right)}{\Gamma\left(\nu + \dfrac{n_1+n_2}{\varrho} + \frac{1}{2} - k\right)}$$

and therefore the resulting series

$$\sum_{n_1,n_2=0}^{\infty} \frac{b_{n_1,n_2}}{(n_1+n_2+1)!} \zeta_1^{-n_1-1} \zeta_2^{-n_2-1}$$

converges uniformly and absolutely for $|\zeta_j| > A^{1/\varrho}$, $j = 1, 2$, where A is the type of the integral function $F(z_1, z_2)$. Hence the change of order of integration and summation is justified under the above conditions.

Now, since the right hand side of (2.1) is analytic function of ζ_1 and ζ_2 under the conditions $|\zeta_j| > A^{1/\varrho}$, $|\arg \zeta_j| < \pi/(2\varrho)$, $j = 1, 2$, therefore, by the principle of analytic continuation, the formula (2.1) is true under the conditions $|\zeta_j| \neq 0$, $|\arg \zeta_j| < \pi/(2\varrho)$, $j = 1, 2$.

If $k = -m = -\frac{1}{2}$, then the above theorem reduces to the following result:

Corollary. *Let* $|\zeta_j| \neq 0$, $|\arg \zeta_j| < \pi/(2\varrho)$, $j = 1, 2$; *and let*

$$F(z_1, z_2) = \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!\, n_2!} z_1^{n_1} z_2^{n_2}$$

be an integral function of two complex variables z_1 *and* z_2 *of order* ϱ $(0 < \varrho < \infty)$, *then, for* $\arg \zeta_1 = \arg \zeta_2$, *we have*

$$J_{n_1,n_2}(\zeta_1, \zeta_2) = \int_0^\infty \int_0^\infty e^{-(t_1\zeta_1 + t_2\zeta_2)^\varrho} (t_1\zeta_1 + t_2\zeta_2)^{\nu\varrho - 2} F(t_1, t_2)\, dt_1\, dt_2$$

$$= \frac{1}{\varrho} \sum_{n_1,n_2=0}^{\infty} a_{n_1,n_2} \frac{\Gamma\left(\nu + \dfrac{n_1+n_2}{\varrho}\right)}{(n_1+n_2+1)!} \zeta_1^{-n_1-1} \zeta_2^{-n_2-1},$$

provided $\mathrm{Re}\,\nu > 0$.

A similar type of result has already been obtained by DŽRBAŠYAN (cfr. [1], p. 261).

Particular case. If we take $\varrho = 1$, $\nu = (3/2) - k$ in Theorem 1, then for $|\zeta_j| \neq 0$, $|\arg \zeta_j| < \pi/2$, $j = 1, 2$ and $\arg \zeta_1 = \arg \zeta_2$, the value of the integral

$$(2.5)\quad \begin{cases} \int_0^\infty \int_0^\infty e^{-\frac{1}{2}(t_1\zeta_1 + t_2\zeta_2)}(t_1\zeta_1 + t_2\zeta_2)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}(t_1\zeta_1 + t_2\zeta_2) F(t_1, t_2)\, dt_1\, dt_2 \\ = \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{(n_1+n_2+1)!} \frac{\Gamma_\times(n_1+n_2+2-k\pm m)}{\Gamma(n_1+n_2+2-2k)} \zeta_1^{-n_1-1}\zeta_2^{-n_2-1}, \end{cases}$$

provided $\operatorname{Re}(2 - k \pm m) > 0$.

If we take $k = -m$ in (2.5), then the value of the integral

$$\int_0^\infty \int_0^\infty e^{-(t_1\zeta_1 + t_2\zeta_2)} F(t_1, t_2)\, dt_1\, dt_2 = \sum_{n_1,n_2=0}^{\infty} a_{n_1,n_2} \zeta_1^{-n_1-1} \zeta_2^{-n_2-1}.$$

3. - Theorem 2. *Let* $|\zeta_j| \neq 0$, $|\arg \zeta_j| < \pi/(2\varrho)$, $j = 1, 2$, *and let*

$$F(z_1, z_2) = \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!\, n_2!} z_1^{n_1} z_2^{n_2},$$

be an integral function of two complex variables z_1 *and* z_2 *of order* $\varrho(0 < \varrho < \infty)$; *then for* $\arg \zeta_1 = \arg \zeta_2$, *we have*

$$(3.1)\quad \begin{cases} P_{n_1,n_2}(\zeta_1, \zeta_2) = \\ \int_0^\infty \int_0^\infty \exp[-\frac{1}{2}(t_1\zeta_1 + t_2\zeta_2)^\varrho - (t_1\zeta_1 + t_2\zeta_2)^{-\varrho}]\,(t_1\zeta_1 + t_2\zeta_2)^{\varrho-2} \cdot \\ \cdot W_{(1/\varrho)(n_1+n_2)+2,m}\left((t_1\zeta_1 + t_2\zeta_2)^\varrho\right) F(t_1, t_2)\, dt_1\, dt_2 \\ = \frac{2K_{2m}(2)}{\varrho} \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{(n_1+n_2+1)!} \zeta_1^{-n_1-1} \zeta_2^{-n_2-1}, \end{cases}$$

provided the change of order of integration and summation is justified.

Proof.

As in Theorem 1, we first take $\operatorname{Re} p > 0$, $\operatorname{Re} a > 0$ to compute the integral

(3.2) $$H_{n_1,n_2}(p,\ a) =$$

$$= \int_0^\infty \int_0^\infty e^{-\frac{1}{2}p(x_1+x_2)^\varrho}\, e^{-a(x_1+x_2)^{-\varrho}}\, (x_1+x_2)^{\varrho-2}\, W_{(1/\varrho)(n_1+n_2)+2,m}\big(p(x_1+x_2)^\varrho\big)\, x_1^{n_1} x_2^{n_2}\, dx_1\, dx_2 .$$

Changing the variables $x_1 = t(1-u)$, $x_2 = tu$, $0 \leqslant u \leqslant 1$, $0 \leqslant t < +\infty$, we obtain

$$H_{n_1,n_2}(p,\ a) =$$

$$= \int_0^\infty \int_0^1 e^{-\frac{1}{2}pt^\varrho}\, e^{-at^{-\varrho}}\, t^{n_1+n_2+\varrho-2}\, W_{(1/\varrho)(n_1+n_2)+2,m}\, (p\,t^\varrho)(1-u)^{n_1}\, u^{n_2}\, \frac{\partial(x_1,\ x_2)}{\partial(t,\ u)}\, dt\, du\ .$$

$$\int_0^\infty \int_0^1 e^{-\frac{1}{2}pt^\varrho} e^{-at^{-\varrho}}\, t^{n_1+n_2+\varrho-1}\, W_{(1/\varrho)(n_1+n_2)+2,m)}\, (p\,t^\varrho)\, (1-u)^{n_1}\, u^{n_2}\, dt\, du\ .$$

Evaluating u-integral with the help of Eulerian integral of the first kind (cfr. [3], p. 212), we obtain

$$H_{n_1,n_2}(p,\ a) =$$

$$= \frac{n_1!\ n_2!}{(n_1+n_2+1)!} \int_0^\infty e^{-\frac{1}{2}pt^\varrho}\, e^{-at^{-\varrho}}\, t^{n_1+n_2+\varrho-1}\, W_{(1/\varrho)(n_1+n_2)+2,m}(p\,t^\varrho)\, dt$$

$$= \frac{n_1!\ n_2!}{\varrho(n_1+n_2+1)!} \int_0^\infty e^{-ax}\, e^{-\frac{1}{2}p/x}\, x^{-(n_1+n_2)/\varrho+2}\, W_{(1/\varrho)(n_1+n_2)+2,m}\, (p/x)\, dx\ .$$

Now using the operational representation due to Goldstein (cfr. [4], p. 107).

$$2p^{-\frac{1}{2}}\, a^{k+\frac{1}{2}}\, K_{2m}\big(2(pa)^{\frac{1}{2}}\big) = L[e^{-\frac{1}{2}p/x}\, W_{k,m}(p/x)x^{-k}]\ , \qquad \mathrm{Re}\, p > 0, \quad \mathrm{Re}\, a > 0,$$

leads to

(3.3) $$H_{n_1,n_2}(p,\ a) = 2\, \frac{n_1!\ n_2!}{\varrho(n_1+n_2+1)!}\, p^{\frac{1}{2}}\, a^{(n_1+n_2)/\varrho+\frac{3}{2}} K_{2m}\big(2(pa)^{\frac{1}{2}}\big),\ \mathrm{Re}\, p>0,\ \mathrm{Re}\, a>0.$$

Let $\arg \zeta_1 = \arg \zeta_2 = \alpha$ and if we denote $p = e^{i\alpha\varrho}$, $a = e^{-i\alpha\varrho}$, where

$\operatorname{Re} p = \operatorname{Re} a = \cos(\alpha\varrho) > 0$, then from (3.1) we obtain

$$
(3.4)\quad \begin{cases}
P_{n_1,n_2}(\zeta_1, \zeta_2) = \\
= \int_0^\infty\int_0^\infty \exp\left[-\tfrac{1}{2}(t_1\zeta_1 + t_2\zeta_2)^\varrho - (t_1\zeta_1 + t_2\zeta_2)^{-\varrho}\right](t_1\zeta_1 + t_2\zeta_2)^{\varrho-2}\cdot \\
\cdot W_{(1/\varrho)(n_1+n_2)+2,m}\left((t_1\zeta_1 + t_2\zeta_2)^\varrho\right)\sum_{n_1,n_2=0}^{\infty}\dfrac{a_{n_1,n_2}}{n_1!\,n_2!}t_1^{n_1}t_2^{n_2}\,dt_1\,dt_2 \\
= e^{i\alpha(\varrho-2)}\sum_{n_1,n_2=0}^{\infty}\dfrac{a_{n_1,n_2}}{n_1!\,n_2!}\int_0^\infty\int_0^\infty \exp\left[-\tfrac{1}{2}p(t_1|\zeta_1|+t_2|\zeta_2|)^\varrho - a(t_1|\zeta_1|+t_2|\zeta_2|)^{-\varrho}\right]\cdot \\
\cdot(t_1|\zeta_1| + t_2|\zeta_2|)^{\varrho-2}W_{(1/\varrho)(n_1+n_2)+2,m}\left(p(t_1|\zeta_1| + t_2|\zeta_2|)^\varrho\right)t_1^{n_1}t_2^{n_2}\,dt_1\,dt_2\,.
\end{cases}
$$

Replacing $t_1|\zeta_1|$ by x_1 and $t_2|\zeta_2|$ by x_2 and evaluating the integrals, we obtain

$$
P_{n_1,n_2}(\zeta_1, \zeta_2) =
$$

$$
= e^{i\alpha(\varrho-2)}\sum_{n_1,n_2=0}^{\infty}\frac{a_{n_1,n_2}}{n_1!\,n_2!}\,\frac{1}{|\zeta_1|^{n_1+1}|\zeta_2|^{n_2+1}}\int_0^\infty\int_0^\infty e^{-\frac{1}{2}p(x_1+x_2)^\varrho}\,e^{-a(x_1+x_2)^{-\varrho}}\,(x_1+x_2)^{\varrho-2}\,.
$$

$$
\cdot W_{(1/\varrho)(n_1+n_2)+2,m}\left(p(x_1+x_2)^\varrho\right)x_1^{n_1}x_2^{n_2}\,dx_1\,dx_2\,,
$$

$$
= \frac{2ae^{i\alpha(\varrho-2)}(pa)^{\frac{1}{2}}K_{2m}(2(pa)^{\frac{1}{2}})}{\varrho}\sum_{n_1,n_2=0}^{\infty}\frac{a_{n_1,n_2}}{(n_1+n_2+1)!}\,\frac{a^{(1/\varrho)(n_1+n_2)}}{|\zeta_1|^{n_1+1}|\zeta_2|^{n_2+1}}
$$

$$
\operatorname{Re} p > 0\,,\qquad \operatorname{Re} a > 0\,,
$$

due to the relation (3.3).

Thus under the given conditions and an appeal to analytic continuation, we obtain

$$
P_{n_1,n_2}(\zeta_1, \zeta_2) = \frac{2K_{2m}(2)}{\varrho}\sum_{n_1,n_2=0}^{\infty}\frac{a_{n_1,n_2}}{(n_1+n_2+1)!}\zeta_1^{-n_1-1}\zeta_2^{-n_2-1}\,,
$$

provided the change of order of integration and summation in (3.4) is justified.

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