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Infinite Integrals Involving G-functions. (**)

§ 1.

Generalisations of the classical LAPLACE transform

$$\varphi(s) = s \int_0^\infty e^{-st} f(t)\, dt\,, \qquad \operatorname{Re} s > 0\,, \tag{1.1}$$

have been given by various mathematicians. Some of them are as follows:

(i) The generalisation due to MEIJER [8] is

$$\varphi(s) = \sqrt{2/\pi} \int_0^\infty \sqrt{st}\, K_m(st)\, f(t)\, dt\,. \tag{1.2}$$

(ii) The MEIJER transform [9] is

$$\varphi(s) = s \int_0^\infty e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(st)\, f(t)\, dt\,. \tag{1.3}$$

(iii) The generalisation due to BOAS [2] is

$$\varphi(s) = \int_0^\infty g(s,\ t)\, f(t)\, dt\,, \tag{1.4}$$

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where $g(s, t)$ in some sense is « nearly » e^{-st}.

(iv) The WHITTAKER transform due to VARMA [11] is

$$\varphi(s) = s \int_0^\infty (2st)^{-1/4} W_{k,m}(st) f(t) \, dt . \tag{1.5}$$

(v) The generalisation due to VARMA [12] is

$$\varphi(s) = s \int_0^\infty (st)^{m-1/2} e^{-1/2 st} W_{k,m}(st) f(t) \, dt . \tag{1.6}$$

(vi) The generalisation due to BHISE [1] is

$$\varphi(s) = \int_0^\infty G^{m+1,0}_{m,m+1} \left[st \middle/ \begin{matrix} p_1 + q_1, \ p_2 + q_2, \ldots, \ p_m + q_m \\ p_1, \ p_2, \ldots, \ p_m, \ \varrho \end{matrix} \right] f(t) \, dt . \tag{1.7}$$

(vii) Recently MAINRA [6] introduced another generalisation in the form

$$\varphi(s) = s \int_0^\infty e^{-1/2 st} (st)^{-\lambda-1/2} W_{k+1/2,m}(st) f(t) \, dt , \tag{1.8}$$

called the generalised MEIJER-LAPLACE transform and is denoted symbolically by

$$\varphi(s; \, k + \tfrac{1}{2}, \, \lambda, \, m) = W[f(t); \, k + \tfrac{1}{2}, \, \lambda, \, m] , \tag{1.9}$$

where $\varphi(s; \, k + \frac{1}{2}, \, \lambda, \, m) = \varphi(s)$, while (1.1) is denoted symbolically by

$$\varphi(s) \doteqdot f(t) . \tag{1.10}$$

It is evident that (1.8) reduces to (1.3) if $\lambda = k$, and to (1.6) when $\lambda = -m$. Further, when $\lambda = k = -m$, (1.8) reduces to (1.1), due to the identity

$$z^{m-1/2} W_{1/2-m,m}(z) = e^{-1/2 z} .$$

We shall call $\varphi(s;\ k+\frac{1}{2},\ \lambda,\ m)$ the image of $f(t)$ or the generalised MEIJER-LAPLACE transform of $f(t)$, and $f(t)$ shall be called the original of $\varphi(s;\ k+\frac{1}{2},\ \lambda,\ m)$.

In this paper we have obtained a theorem for this transform and have applied this in evaluating a few infinite integrals. Most of the integrals evaluated involve MEIJER's G-function. The importance of the results obtained lies due to the fact that a great many of the special functions, appearing in applied mathematics, are particular cases of G-function, and so the results obtained are quite general. From these a number of results can be deduced for BESSEL, LEGENDRE, LAGUERRE, WHITTAKER functions, their combinations and other related functions as particular cases.

§ 2.

Theorem I. *If*

$$W[f(t);\ k+\tfrac{1}{2},\ \lambda,\ m] = \varphi(s) \tag{2.1}$$

and

$$W[(1/\sqrt{t})\,\varphi(1/t);\ k+\tfrac{1}{2},\ \lambda,\ m] = g(s)\,, \tag{2.2}$$

then

$$W[f(t^2);\ k+\tfrac{1}{2},\ \lambda,\ m] = \frac{2^{k-\lambda+1}}{s\sqrt{\pi}}\, g(s^2/4;\ (k/2)+\tfrac{1}{2},\ \lambda/2\,,\ m/2)\,, \tag{2.3}$$

provided $\mathrm{Re}\, s > s_0 > 0$ *and the generalised Meijer-Laplace transform of* $|f(t)|$ *and* $|f(t^2)|$ *exist.*

Proof. From (1.8), we have

$$\varphi(s) = s\int_0^\infty e^{-\frac{1}{2}sx}\,(sx)^{-\lambda-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(sx)\, f(x)\, \mathrm{d}x\,.$$

Therefore,

$$\varphi(1/t) = (1/t)\int_0^\infty e^{-\frac{1}{2}x/t}\,(x/t)^{-\lambda-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(x/t)\, f(x)\, \mathrm{d}x$$

and

$$\frac{1}{\sqrt{t}}\,\varphi(1/t) = \frac{1}{t^{3/2}}\int_0^\infty e^{-\frac{1}{2}x/t}\,(x/t)^{-\lambda-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(x/t)\, f(x)\, \mathrm{d}x\,.$$

Also:

$$g(s) = s\int_0^\infty e^{-\frac{1}{2}st}\,(st)^{-\lambda-\frac{1}{2}}\,W_{k+\frac{1}{2},m}(st)\,\frac{1}{\sqrt{t}}\,\varphi(1/t)\,\mathrm{d}t$$

$$= s\int_0^\infty e^{-\frac{1}{2}st}\,(st)^{-\lambda-\frac{1}{2}}\,W_{k+\frac{1}{2},m}(st)\,\mathrm{d}t$$

$$= t^{-3/2}\int_0^\infty e^{-\frac{1}{2}x/t}\,(x/t)^{-\lambda-\frac{1}{2}}\,W_{k+\frac{1}{2},m}(x/t)\,f(x)\,\mathrm{d}x.$$

Changing the order of integration, which is justifiable, we get

$$g(s) = s\int_0^\infty (sx)^{-\lambda-\frac{1}{2}}\,f(x)\,[\int_0^\infty t^{-3/2}\,e^{-\frac{1}{2}(st+(x/t))}\,W_{k+\frac{1}{2},m}(st)\,W_{k+\frac{1}{2},m}(x/t)\,\mathrm{d}t]\,\mathrm{d}x\,.$$

Evaluating the integral with the help of a known formula [5], namely

$$\int_0^\infty x^{\varrho-1}\exp\left[\tfrac{1}{2}\left(\frac{x}{\alpha}+\frac{\beta}{x}\right)\right]W_{k,\mu}(x/\alpha)\,W_{\lambda,\nu}(\beta/x)\,\mathrm{d}x =$$

$$= \beta^{\varrho}\,G^{4,0}_{2,4}\left[\frac{\beta}{\alpha}\Big/\begin{matrix}1-k,\ 1-\lambda-\varrho\\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu,\ \frac{1}{2}-\varrho+\nu,\ \frac{1}{2}-\varrho-\nu\end{matrix}\right],$$

where $\operatorname{Re}\varrho > 0$ and $\operatorname{Re}\beta > 0$, we get

$$g(s) = s\int_0^\infty (sx)^{-\lambda-\frac{1}{2}}\,x^{-\frac{1}{2}}\,G^{4,0}_{2,4}\left[sx\Big/\begin{matrix}\frac{1}{2}-k,\ 1-k\\ \frac{1}{2}+m,\ \frac{1}{2}-m,\ 1+m,\ 1-m\end{matrix}\right]f(x)\,\mathrm{d}x\,,$$

provided $\operatorname{Re}s > 0$ and $\operatorname{Re}x > 0$. On writing $s^2/4$ for s and x^2 for x, we get

$$g(s^2/4) = \frac{s^2}{4}\int_0^\infty (sx/2)^{-2\lambda-1}\,(sx)\,G^{4,0}_{2,4}\left[\frac{s^2x^2}{4}\Big/\begin{matrix}-k,\ \frac{1}{2}-k\\ m,\ -m,\ \frac{1}{2}+m,\ \frac{1}{2}-m\end{matrix}\right]f(x^2)\,\mathrm{d}x\,.$$

By a known relation [4], namely

$$G_{2,4}^{4,0}\left[x \,\middle/\, \begin{matrix} a,\ a+\frac{1}{2} \\ h+c,\ h-c,\ h+c+\frac{1}{2},\ h-c+\frac{1}{2} \end{matrix}\right] = \sqrt{\pi}\, 2^{-k}\, x^{h-1/4}\, e^{-\sqrt{x}}\, W_{k,2c}(2\sqrt{x}),$$

where $k = \frac{1}{2} + 2h - 2c$, we get

$$g(s^2/4) = \frac{s\sqrt{\pi}}{2^{1+2k-2\lambda}} \int_0^\infty e^{-\frac{1}{2}sx} (sx)^{-2\lambda-\frac{1}{2}}\, W_{2k+\frac{1}{2},2m}(sx)\, f(x^2)\, dx\,,$$

i.e.

$$g(s^2/4;\ k+\tfrac{1}{2},\ \lambda,\ m) = \frac{2\sqrt{\pi}}{2^{1+2k-2\lambda}}\, W[f(t^2);\ 2k+\tfrac{1}{2},\ 2\lambda,\ 2m]\,.$$

Writing $k/2$, $\lambda/2$ and $m/2$ for k, λ and m respectively, we get

$$(2.4)\qquad g(s^2/4;\ (k/2)+\tfrac{1}{2},\ \lambda/2,\ m/2) = \frac{s\sqrt{\pi}}{2^{k-\lambda+1}}\, W[f(t^2);\ k+\tfrac{1}{2},\ \lambda,\ m].$$

Hence the result.

The inversion in the order of integration is justified by DE LA VALLÉE POUSSIN's theorem [3] under the conditions stated in the Theorem.

If $\lambda = k = -m$, the Theorem reduces to a well-known theorem in LAPLACE transform (1.1) given below [7]:

Corollary. *If*

$$\varphi(s) \doteqdot f(t)$$

and

$$g(s) \doteqdot (1/\sqrt{t})\, \varphi(1/t)\,,$$

then

$$g(s^2/4) \doteqdot (\sqrt{\pi}/2)\, f(t^2)\,,$$

provided $\mathrm{Re}\, s \geqslant s_0 > 0$ *and the Laplace transform of* $|f(t)|$ *and* $|f(t^2)|$ *exist.*

For illustrating the Theorem, the following example is worth mention.

Let

$$f(t) = t^{-\varrho} G^{l,n}_{p,q}\left[ht \Big/ \begin{matrix}\alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}\right].$$

Then, we have [10]

$$\varphi(s) = s^{\varrho} G^{l,n+2}_{p+2,q+1}\left[\frac{h}{s} \Big/ \begin{matrix}\lambda + \varrho \pm m, \ \lambda + \varrho - m, \ \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q, \ \lambda + k + \varrho\end{matrix}\right],$$

where $\operatorname{Re}(\beta_j - \varrho + 1 - \lambda \pm m) > 0$, $j = 1, 2, \dots, l$, $|\arg(s/h)| < (l + n - \frac{1}{2} p - \frac{1}{2} q - \frac{1}{2}) \pi$ and $p + q + 1 < 2(l + n)$. Hence:

$$\frac{1}{\sqrt{t}} \varphi(1/t) = t^{-\varrho - \frac{1}{2}} G^{l,n+2}_{p+2,q+1}\left[ht \Big/ \begin{matrix}\lambda + \varrho + m, \ \lambda + \varrho - m, \ \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q, \ \lambda + \varrho + k\end{matrix}\right].$$

Thus we have

$$g(s) = s^{\frac{1}{2} + \varrho} G^{l,n+4}_{p+4,q+2}\left[\frac{h}{s} \Big/ \begin{matrix}\lambda + \frac{1}{2} + \varrho + m, \ \lambda + \frac{1}{2} + \varrho - m, \ \lambda + \varrho + m, \ \lambda + \varrho, \ \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q, \ \lambda + \varrho + k, \ \lambda + \varrho + k + \frac{1}{2}\end{matrix}\right],$$

provided that $\operatorname{Re}(\beta_j - \varrho + \frac{1}{2} - \lambda \pm m) > 0$, $j = 1, 2, \dots, l$, $|\arg(s/h)| < (l + n - \frac{1}{2} p - \frac{1}{2} q - \frac{1}{2}) \pi$, $\operatorname{Re} s > 0$ and $p + q + 1 < 2(l + n)$.

Now, Theorem 1 gives that

$$W\left[t^{-2\varrho} G^{l,n}_{p,q}\left[\beta t^2 \Big/ \begin{matrix}\alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}\right]; \ k + \tfrac{1}{2}, \ \lambda, \ m\right] = \frac{s^{2\varrho}}{\sqrt{\pi}\, 2^{2\varrho + \lambda - k}} G^{l,n+4}_{p+4,q+2}\left[\frac{4\beta}{s^2} \Big/ \begin{matrix}(\lambda/2) + \frac{1}{2} + \varrho + (m/2), (\lambda/2) + \frac{1}{2} + \varrho - (m/2), (\lambda/2) + \varrho + (m/2), (\lambda/2) + \varrho - (m/2), \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q, \ (\lambda/2) + (k/2) + \varrho, \ (\lambda/2) + (k/2) + \frac{1}{2} + \varrho\end{matrix}\right],$$

provided that $\operatorname{Re}(\beta_j - \varrho - \frac{1}{2} - \lambda \pm m) > 0$, $j = 1, 2, \dots, l$, $|\arg(s/h)| < (l + n - \frac{1}{2} p - \frac{1}{2} q - \frac{1}{2}) \pi$, $\operatorname{Re} s > 0$ and $p + q + 1 < 2(l + n)$,

$$(2.5) \quad \begin{cases} s \displaystyle\int_0^{\infty} (st)^{-\lambda - \frac{1}{2}} e^{-\frac{1}{2} st} W_{k + \frac{1}{2}, m}(st)\, t^{-2\varrho} G^{l,n}_{p,q}\left[\beta t^2 \Big/ \begin{matrix}\alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}\right] dt = \\[2ex] \quad = \dfrac{s^{2\varrho}}{\sqrt{\pi}\, 2^{2\varrho + \lambda - k}} G^{l,n+4}_{p+4,q+2}\left[\dfrac{4\beta}{s^2} \Big/ \begin{matrix}(\lambda/2) + \frac{1}{2} + \varrho + (m/2), (\lambda/2) + \frac{1}{2} + \varrho - (m/2), (\lambda/2) + \varrho + (m/2), (\lambda/2) + \varrho - (m/2), \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q, \ (\lambda/2) + \varrho + (k/2), \ \frac{1}{2} + (\lambda/2) + \varrho + (k/2)\end{matrix}\right]. \end{cases}$$

Thus Theorem 1 is helpful in evaluating infinite integral involving MEIJER's G-function. Some interesting particular cases are given below:

(i) If $\varrho = 0$, we get

$$(2.6)\quad \begin{cases} s\int_0^\infty (st)^{-\lambda-\frac{1}{2}}\, e^{-\frac{1}{2}st}\, W_{k+\frac{1}{2},m}(st)\, G_{p,q}^{l,n}\left[\beta t^2 \middle/ \begin{matrix}\alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q\end{matrix}\right] dt = \\ \qquad = \dfrac{1}{\sqrt{\pi}\; 2^{\lambda-k}}\, G_{p+4,q+2}^{l,n+4}\left[\dfrac{4\beta}{s^2} \middle/ \begin{matrix}(\lambda/2)+\frac{1}{2}+(m/2),\ (\lambda/2)+\frac{1}{2}-(m/2),\ (\lambda/2)+(m/2),\ (\lambda/2)-(m/2),\ \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q,\ (\lambda/2)+(k/2),\ \frac{1}{2}+(\lambda/2)+(k/2)\end{matrix}\right]. \end{cases}$$

(ii) When $\lambda = k$, we get

$$(2.7)\quad \begin{cases} s\int_0^\infty e^{-\frac{1}{2}st}\, (st)^{-k-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(st)\, t^{-2\varrho}\, G_{p,q}^{l,n}\left[\beta t^2 \middle/ \begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q\end{matrix}\right] dt = \\ \qquad = \dfrac{s^{2\varrho}}{\sqrt{\pi}\, 2^{2\varrho}}\, G_{p+4,q+2}^{l,n+4}\left[\dfrac{4\beta}{s^2} \middle/ \begin{matrix}(k/2)+\frac{1}{2}+\varrho+(m/2),\ (k/2)+\frac{1}{2}+\varrho-(m/2),\ (k/2)+\varrho-(m/2),\ \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q,\ k+\varrho,\ \frac{1}{2}+k+\varrho\end{matrix}\right]. \end{cases}$$

(iii) When $\lambda = k$ and $\varrho = 0$, we get the following:

$$(2.8)\quad \begin{cases} s\int_0^\infty e^{-\frac{1}{2}st}\, (st)^{-k-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(st)\, G_{p,q}^{l,n}\left[\beta t^2 \middle/ \begin{matrix}\alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q\end{matrix}\right] dt = \\ \qquad = \dfrac{1}{\sqrt{\pi}}\, G_{p+4,q+2}^{l,n+4}\left[\dfrac{4\beta}{s^2} \middle/ \begin{matrix}(k/2)+\frac{1}{2}+(m/2),\ (k/2)+\frac{1}{2}-(m/2),\ (k/2)+(m/2),\ (k/2)-(m/2),\ \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q,\ k,\ \frac{1}{2}+k\end{matrix}\right]. \end{cases}$$

(iv) When $\lambda = -m$, we get

(2.9)
$$\begin{cases} s\int_0^\infty e^{-\frac{1}{2}st}(st)^{m-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(st)\, t^{-2\varrho}\, G_{p,q}^{l,n}\left[\beta t^2 \Big/ \begin{matrix}\alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}\right] dt = \\ = \dfrac{s^{2\varrho}}{\sqrt{\pi}\, 2^{2\varrho-m-k}}\, G_{p+4,q+2}^{l,n+4}\left[\dfrac{4\beta}{s^2} \Big/ \right. \\ \left. \begin{matrix}\frac{1}{2}+\varrho,\ \frac{1}{2}+\varrho-(m/2),\ \varrho,\ \varrho-(m/2),\ \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q,\ \varrho-(m/2)+(k/2),\ \frac{1}{2}+\varrho+(k/2)-(m/2)\end{matrix}\right]. \end{cases}$$

(v) When $\lambda = -m$ and $\varrho = 0$, we have

(2.10)
$$\begin{cases} s\int_0^\infty e^{-\frac{1}{2}st}(st)^{m-\frac{1}{2}}\, W_{k+\frac{1}{2},m}(st)\, G_{p,q}^{l,n}\left[\beta t^2 \Big/ \begin{matrix}\alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}\right] dt = \\ = \dfrac{1}{\sqrt{\pi}\, 2^{-m-k}}\, G_{p+4,q+2}^{l,n+4}\left[\dfrac{4\beta}{s^2} \Big/ \begin{matrix}\frac{1}{2},\ \frac{1}{2}-(m/2),\ 0,\ -(m/2),\ \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q,\ -(m/2)+(k/2),\ \frac{1}{2}+(k/2)-(m/2)\end{matrix}\right]. \end{cases}$$

(vi) When $\lambda = k = -m$, we get

(2.11)
$$\begin{cases} s\int_0^\infty e^{-st}\, t^{-2\varrho}\, G_{p,q}^{l,n}\left[\beta t^2 \Big/ \begin{matrix}\alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}\right] dt = \\ = \dfrac{s^{2\varrho}}{\sqrt{\pi}\, 2^{2\varrho}}\, G_{p+2,q}^{l,n+2}\left[\dfrac{4\beta}{s^2} \Big/ \begin{matrix}\frac{1}{2}+\varrho,\ \varrho,\ \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}\right]. \end{cases}$$

(vii) When $\lambda = k = -m$ and $\varrho = 0$, we get

(2.12)
$$\begin{cases} s\int_0^\infty e^{-st}\, G_{p,q}^{l,n}\left[\beta t^2 \Big/ \begin{matrix}\alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}\right] dt = \\ = \dfrac{1}{\sqrt{\pi}}\, G_{p+2,q}^{l,n+2}\left[\dfrac{4\beta}{s^2} \Big/ \begin{matrix}0,\ \frac{1}{2},\ \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}\right]. \end{cases}$$

If we write k in place of $k+\frac{1}{2}$ in (2.5), we get

$$(2.13)\quad \begin{cases} \displaystyle\int_0^\infty t^{-\lambda-\frac{1}{2}-2\varrho}\, e^{-\frac{1}{2}st}\, W_{k,m}(st)\, G_{p,q}^{l,n}\left[\beta t^2 \middle/ \begin{matrix}\alpha_1, \ldots, \alpha_p\\ \beta_1, \ldots, \beta_q\end{matrix}\right] dt = \\[2ex] \displaystyle = \frac{s^{2\varrho+\lambda-\frac{1}{2}}}{\sqrt{\pi}\, 2^{2\varrho+\lambda+\frac{1}{2}-k}}\, G_{p+4,q+2}^{l,n+4}\left[\frac{4\beta}{s^2} \middle/ \right. \\[2ex] \left. \begin{matrix} (\lambda/2)+\frac{1}{2}+\varrho \pm (m/2),\ (\lambda/2)+\varrho \pm (m/2),\ \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q,\ (\lambda/2)+\varrho+(k/2)-(1/4),\ (\lambda/2)+\varrho+(k/2)+(1/4) \end{matrix}\right]. \end{cases}$$

Now, using the formula [5]

$$W_{k,1/4}(st) = 2^{-k}\,(2st)^{1/4}\, D_{2k-\frac{1}{2}}(2\sqrt{st})$$

in (2.13), we get

$$(2.14)\quad \begin{cases} \displaystyle\int_0^\infty t^{\lambda-1/4-2\varrho}\, D_{2k-\frac{1}{2}}(2\sqrt{st})\, e^{-\frac{1}{2}st}\, G_{p,q}^{l,n}\left[\beta t^2 \middle/ \begin{matrix}\alpha_1, \ldots, \alpha_p\\ \beta_1, \ldots, \beta_q\end{matrix}\right] dt = \\[2ex] \displaystyle = \frac{s^{2\varrho+\lambda-(3/4)}}{\sqrt{\pi}\ 2^{2\varrho+\lambda-2k+(3/4)}}\, G_{p+4,q+2}^{l,n+4}\left[\frac{4\beta}{s^2} \middle/ \right. \\[2ex] \left. \begin{matrix} (\lambda/2)+\varrho+\frac{1}{2} \pm (1/8),\ (\lambda/2)+\varrho \pm (1/8),\ \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q,\ (\lambda/2)+\varrho+(k/2)-(1/4),\ (\lambda/2)+\varrho+(k/2)+(1/4) \end{matrix}\right]. \end{cases}$$

Again, using another formula [5] namely

$$W_{0,m}(st) = (st/\pi)^{\frac{1}{2}}\, K_m(st/2)\,,$$

in (2.13), we get

$$(2.15)\quad \begin{cases} \displaystyle\int_0^\infty t^{-\lambda-2\varrho}\, e^{-\frac{1}{2}st}\, K_m(st/2)\, G_{p,q}^{l,n}\left[\beta t^2 \middle/ \begin{matrix}\alpha_1, \ldots, \alpha_p\\ \beta_1, \ldots, \beta_q\end{matrix}\right] dt = \\[2ex] \displaystyle = \frac{s^{2\varrho+\lambda-1}}{2^{2\varrho+\lambda+\frac{1}{2}}}\, G_{p+4,q+2}^{l,n+4}\left[\frac{4\beta}{s^2} \middle/ \begin{matrix} (\lambda/2)+\frac{1}{2}+\varrho \pm (m/2),\ (\lambda/2)+\varrho \pm (m/2),\ \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q,\ (\lambda/2)+\varrho-(1/4),\ (\lambda/2)+(1/4)+\varrho \end{matrix}\right]. \end{cases}$$

Further, using the formula [5

$$W_{m+k+\frac{1}{2},m}(st) = (-1)^k\, k!\, (st)^{m+\frac{1}{2}}\, e^{-\frac{1}{2}st}\, L_k^{2m}(st)$$

in (2.13), we obtain

(2.16)
$$\left\{\begin{aligned} &(-1)^k\, k! \int_0^\infty t^{m-\lambda-2\varrho}\, e^{-st}\, L_k^{2m}(st)\, G_{p,q}^{l,n}\left[\beta t^2 \Big/ \begin{matrix}\alpha_1, \dots, \alpha_p\\ \beta_1, \dots, \beta_q\end{matrix}\right] dt = \\ &= \frac{s^{2\varrho-1+\lambda-m}}{\sqrt{\pi}\, 2^{2\varrho+\lambda-m-k}}\, G_{p+4,q+2}^{l,n+4}\left[\frac{4\beta}{s^2} \Big/ \right. \\ &\left.\begin{matrix}(\lambda/2)+\frac{1}{2}+\varrho \pm (m/2),\ (\lambda/2)+\varrho \pm (m/2),\ \alpha_1, \dots, \alpha_p\\ \beta_1, \dots, \beta_q,\ (\lambda/2)+\varrho+(m/2)+(k/2),\ (\lambda/2)+\varrho+\frac{1}{2}+(m/2)+(k/2)\end{matrix}\right].\end{aligned}\right.$$

Using another relation [4]

$$e^{-\frac{1}{2}st}\, W_{k,m}(st) = \sqrt{st/\pi}\, 2^{k-\frac{1}{2}}\, G_{2,4}^{4,0}\left[\frac{s^2 t^2}{4} \Big/ \begin{matrix}(1/4)-(k/2),\ (3/4)-(k/2)\\ \frac{1}{2} \pm (m/2),\ \pm (m/2)\end{matrix}\right]$$

in (2.13), we get

$$2^{k-\frac{1}{2}}\sqrt{s/\pi}\int_0^\infty t^{-\lambda-2\varrho}\, G_{2,4}^{4,0}\left[\frac{s^2 t^2}{4} \Big/ \begin{matrix}(1/4)-\frac{1}{2}k,\ (3/4)-\frac{1}{2}k\\ \frac{1}{2} \pm (m/2),\ \pm (m/2)\end{matrix}\right] G_{p,q}^{l,n}\left[\beta t^2 \Big/ \begin{matrix}\alpha_1, \dots, \alpha_p\\ \beta_1, \dots, \beta_q\end{matrix}\right] dt =$$

$$= \frac{s^{2\varrho+\lambda-\frac{1}{2}}}{\sqrt{\pi}\, 2^{2\varrho+\lambda+\frac{1}{2}-k}}\, G_{p+4,q+2}^{l,n+4}\left[\frac{4\beta}{s^2} \Big/ \right.$$

$$\left.\begin{matrix}(\lambda/2)+\frac{1}{2}+\varrho \pm (m/2),\ (\lambda/2)+\varrho \pm (m/2),\ \alpha_1, \alpha_2, \dots, \alpha_p\\ \beta_1, \dots, \beta_q,\ (\lambda/2)+\varrho+(k/2)-(1/4),\ (1/4)+(\lambda/2)+\varrho+(k/2)\end{matrix}\right],$$

i.e.

$$\int_0^\infty G_{2,4}^{4,0}\left[\frac{s^2}{4} u \Big/ \begin{matrix}-(1/4)-\varrho-(\lambda/2)-(k/2),\ (1/4)-(k/2)-(\lambda/2)-\varrho\\ -(\lambda/2)-\varrho \pm (m/2),\ -\frac{1}{2}-(\lambda/2)-\varrho \pm (m/2)\end{matrix}\right].$$

(2.17)
$$\cdot\, G_{p,q}^{l,n}\left[\beta u^2 \Big/ \begin{matrix}\alpha_1, \dots, \alpha_p\\ \beta_1, \dots, \beta_q\end{matrix}\right] dx =$$

$$= \frac{4}{s^2} G^{l,n+4}_{p+4,q+2}\left[\frac{4\beta}{s^2} \middle/ \begin{matrix} (\lambda/2)+\frac{1}{2}+\varrho \pm (m/2),\ (\lambda/2)+\varrho \pm (m/2),\ \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q,\ -(1/4)+\lambda+\varrho+(k/2),\ (1/4)+(\lambda/2)+\varrho+(k/2) \end{matrix}\right].$$

This is a well-known result given in [5], p. 422.

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