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Approximate Solution of the Exterior Dirichlet Problem and the Calculation of Electrostatic Capacity. (**)

1. - The Exterior Dirichlet Problem.

In Euclidean n-space E^n, let the open set R_1 be bounded, contain the origin, and be homeomorphic to the interior of the unit sphere. Let the boundary of R_1 be S and let φ be defined and continuous on S. Set $R = [E^n - (R_1 \cup S)]$. Then the exterior DIRICHLET problem is that of finding a function u on $R \cup S$ which has the following properties:

(i) on R, u is a solution of LAPLACE's equation;

(ii) on S, $u \equiv \varphi$;

(iii) u is continuous on $R \cup S$; and

(iv) u is bounded on $R \cup S$ if $n = 2$, while if $n > 2$ there exists a constant M such that for all points $P \in (R \cup S)$ one has

$$| u(P) | \leqslant (M \div \overline{OP}) .$$

Under weak assumptions on the structure of S, it is known that the above problem has a unique solution (PETROVSKY [1], pp. 189-192) and it is only

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with such problems that we shall be concerned. Moreover, since it is only under exceptional circumstances that the solution can be given constructively (BICKLEY [2]), we shall be interested here in describing a digital computer technique for approximating u and related quantities.

Property iv, above, will require that the cases $n = 2$ and $n > 2$ be treated differently. However, it will be shown that the numerical method for $n = 2$ can be modified almost trivially so that it is applicable to the case $n > 2$. Special attention will be given to the case $n = 3$ and application will be made to the estimation of electrostatic capacity (PÓLYA and SZEGÖ [3], Ch. III).

2. - The Exterior Dirichlet Problem in E^2.

The approximation method to be developed here is the natural one of applying an inversion mapping to the given problem, then solving the resulting interior problem numerically, and finally reinverting to yield an approximation of the given exterior problem. For this purpose, the notation $u = u(x, y)$ on $R \cup S$, $\varphi = \varphi(x, y)$ on S and $\Delta u = u_{xx} + u_{yy} = 0$ on R will be useful. Our method will be called Method E (for Exterior) and is formalized as follows.

Method E.

Step 1. Construct S^i and R^i, where $S \to S^i$ and $R \to R^i$ under the inversion mapping

$$\xi = \frac{x}{x^2 + y^2}, \qquad \eta = \frac{y}{x^2 + y^2}, \qquad (x^2 + y^2 \neq 0). \tag{2.1}$$

Since (2.1) implies

$$x = \frac{\xi}{\xi^2 + \eta^2}, \qquad y = \frac{\eta}{\xi^2 + \eta^2}, \qquad (\xi^2 + \eta^2 \neq 0), \tag{2.2}$$

set

$$u(x, y) = u\left[\frac{\xi}{\xi^2 + \eta^2}, \frac{\eta}{\xi^2 + \eta^2}\right] = v(\xi, \eta), \tag{2.3}$$

$$\varphi(x, y) = \varphi\left[\frac{\xi}{\xi^2 + \eta^2}, \frac{\eta}{\xi^2 + \eta^2}\right] = \omega(\xi, \eta). \tag{2.4}$$

Then it is known (PETROVSKY [1], pp. 189-192) that the exterior DIRICHLET problem defined in Section 1 is equivalent to the following interior DIRICHLET problem:

Find $v(\xi, \eta)$ on $R^i \cup S^i$ which has the properties

(a) on R^i, v is a solution of the LAPLACE equation

$$v_{\xi\xi} + v_{\eta\eta} = 0; \tag{2.5}$$

(b) on S^i,

$$v(\xi, \eta) = \omega(\xi, \eta); \tag{2.6}$$

(c) $v(\xi, \eta)$ is continuous on $R^i \cup S^i$.

Step 2. Fix $(\bar{\xi}, \bar{\eta}) \in R^i \cup S^i$ and for $h > 0$ determine the planar grid points $(\bar{\xi} + ph, \bar{\eta} + qh)$; $p = 0, \pm 1, \pm 2, \ldots$; $q = 0, \pm 1, \pm 2, \ldots$. The set of all horizontal and all vetical lines through the planar grid points is called a planar lattice. Now, let R_h^i denote those planar grid points which are also elements of R^i and let S_h^i denote the points of intersection of the planar lattice and S. R_h^i is called the set of interior lattice points associated with $R \cup S$ while S_h^i is called the set of boundary lattice points associated with $R \cup S$. For a point $(\xi, \eta) \in R_h^i$, we call the four points $(\xi + h_1, \eta)$, $(\xi, \eta + h_2)$, $(\xi - h_3, \eta)$, $(\xi, \eta - h_4)$, which are points of $R_h^i \cup S_h^i$ and which are nearest to (ξ, η) in the east, north, west and south directions, respectively, the neighbors of (ξ, η).

If then R_h^i consists of m points and S_h^i consists of n points, number the points of R_h^i in a one-to-one fashion with the integers $1, 2, \ldots, m$ and the points of S_h^i in a one-to-one fashion with the integers $m + 1, m + 2, \ldots, m + n$. Further, if W is a function defined on $R_h^i \cup S_h^i$ and if point (ξ, η) has been numbered k, we shall use the subscript notation $W(\xi, \eta) = W_k$.

Step 3. At each point $(\xi, \eta) \in S_h^i$, set

$$V(\xi, \eta) = \omega(\xi, \eta). \tag{2.7}$$

Thus, each V_k, for $k = m + 1, m + 2, \ldots, m + n$, is known.

Step 4. At each point $(\xi, \eta) \in R_h^i$, set down in subscript notation the LAPLACE difference analogue

$$\begin{cases} -2\left[\dfrac{1}{h_1 h_3} + \dfrac{1}{h_2 h_4}\right] V(\xi, \eta) + \dfrac{2}{h_1(h_1 + h_3)} V(\xi + h_1, \eta) + \dfrac{2}{h_2(h_2 + h_4)} V(\xi, \eta + h_2) + \\ \qquad + \dfrac{2}{h_3(h_1 + h_3)} V(\xi - h_3, \eta) + \dfrac{2}{h_4(h_2 + h_4)} V(\xi, \eta - h_4) = 0, \end{cases} \tag{2.8}$$

where $(\xi + h_1, \eta)$, $(\xi, \eta + h_2)$, $(\xi - h_3, \eta)$, $(\xi, \eta - h_4)$ are the four neighbors of (ξ, η), and where the known values determined in Step 3 are inserted whenever possible. There results then a linear algebraic system of m equations in $V_1, V_2, \ldots, V_m$. Solve this system.

Step 5. Let the V_k $(k = 1, 2, \ldots, m, m+1, \ldots, m+n)$ generated in Steps 3 and 4 represent the approximation on $R_h^i \cup S_h^i$ of the $v(\xi, \eta)$ which satisfies (2.5) and (2.6).

Step 6. Map each point $(\xi, \eta) \in R_h^i \subset S_h^i$ into $R \cup S$ by means of (2.2). Further, if (ξ, η) was numbered k, let its map (x, y) also be numbered k and set

$$V_k = V(\xi, \eta) = V\left[\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right] = U(x, y) = U_k . \tag{2.9}$$

Finally, on the points of $R \cup S$ numbered $1, 2, \ldots, m, m+1, \ldots, m+n$, let the U_k, $k = 1, 2, \ldots, m+n$ defined by (2.9) be the approximations to the solution $u(x, y)$ of the given exterior DIRICHLET problem, and the method is complete.

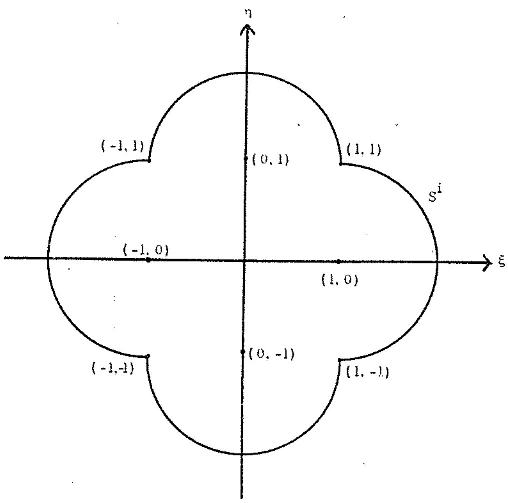

Diagram 2.1

One of the great advantages of Method E is that it is founded on an interior method, and because of the equivalence of the exterior and interior problems described, all the theoretical support for the approximation V_k of v extends to the approximation U_k of u. Thus (GREENSPAN, [4]) U_k $(k = 1, 2, \ldots, m+n)$, exists, is unique, and converges to $u(x, y)$ ad $h \to 0$ provided $u(x,y) \in C^2(R \cup S)$.

To demonstrate the ease with which Method E can be applied, the following illustrative example, typical of those run on the CDC 1604 at the University of Wisconsin, is now presented.

Let S be the unit square with vertices $\left[\frac{1}{2}, \frac{1}{2}\right]$, $\left[-\frac{1}{2}, \frac{1}{2}\right]$, $\left[-\frac{1}{2}, -\frac{1}{2}\right]$, $\left[\frac{1}{2}, -\frac{1}{2}\right]$. Let $\varphi(x, y) = (x^2 - y^2)(x^2 + y^2)^{-2}$, and consider the associated exterior DIRICHLET problem defined in Section 1. Under transformation (2.2), each straight line segment of S maps into a semi-circle in the $\xi - \eta$ plane and S^i is as shown in Diagram 2.1. From (2.4), it follows that

$$\omega(\xi, \eta) = \xi^2 - \eta^2 . \qquad (2.10)$$

For the interior problem then defined by (2.5) and (2.6), set $(\bar{\xi}, \bar{\eta}) = (0, 0)$ and $h = .02$. Step 4 of Method E yielded a system of 25,649 equations in 25,649 unknowns, which, by symmetry considerations about both the ξ and η axes reduced to 6512 equations in 6512 unknowns. The resulting system was solved

TABLE 1.

ξ	η	x	y	Computed value U	Exact value u
.0000	.0200	.0000	50.0000	.00040005	.00040000
.0400	.7600	.0691	1.3122	− .57600067	− .57600000
.1400	.1800	2.6923	3.4615	− .01280012	− .01280000
.2600	1.5800	.1014	.6162	− 2.42880003	− 2.42880000
.4200	.3400	1.4384	1.1644	.06080005	.06080000
.5000	.0200	1.9968	.0799	.24960062	.24960000
.6400	.2600	1.3412	.5448	.34200036	.34200000
1.1800	.3000	.7960	.2024	1.30240017	1.30240000

with a successive over-relaxation factor of 1.94. The number of iterations was 287 and the running time was 19 minutes 49 seconds. The numerical results, when compared with the exact solution $v(\xi, \eta) = \xi^2 - \eta^2$ (which was known for the given problem) were correct to at least six decimal places. For typical final results consult Table 1.

Note also that if one wished to approximate u at certain fixed points of R, then it would be desirable to generalize Method E by considering a non-uniform grid in Step 2. Of course this can be done in the usual fashion without any loss of theoretical support.

Note finally that Method E is formulated around an interior method which is particularly suitable when S^i is irregular, a complexity arising in the present discussion from the application of inversion to S. Other interior methods suitable to irregular S^i can be used in a similar way to construct new exterior methods.

3. - The Exterior Problem in E^n.

Since the treatment of the exterior problem for E^n will be a direct generalization of that for E^3, our remarks will be directed primarily to E^3. For this purpose the notation $u = u(x, y, z)$ on $R \cup S$, $\varphi = \varphi(x, y, z)$ on S, and $\Delta u \equiv \equiv u_{xx} + u_{yy} + u_{zz} = 0$ on R will be convenient. Under the inversion mapping

$$(3.1) \quad \xi = \frac{x}{x^2+y^2+z^2}, \quad \eta = \frac{y}{x^2+y^2+z^2}, \quad \nu = \frac{z}{x^2+y^2+z^2}, \quad x^2+y^2+z^2 \neq 0$$

or, equivalently,

$$(3.2) \quad x = \frac{\xi}{\xi^2+\eta^2+\nu^2}, \quad y = \frac{\eta}{\xi^2+\eta^2+\nu^2}, \quad z = \frac{\nu}{\xi^2+\eta^2+\nu^2}, \quad \xi^2+\eta^2+\nu^2 \neq 0,$$

let $R \to R^i$, $S \to S^i$. Define

$$(3.3) \qquad \Phi(\xi, \eta, \nu) = \frac{\varphi(x, y, z)}{\sqrt{\xi^2+\eta^2+\nu^2}},$$

$$(3.4) \qquad v(\xi, \eta, \nu) = \frac{u(x, y, z)}{\sqrt{\xi^2+\eta^2+\nu^2}},$$

where x, y, z, ξ, η, ν are related by (3.1), or equivalently, by (3.2). Then (PETROVSKY [1], p. 190), $v(\xi, \eta, \nu)$ is the solution of the interior DIRICHLET problem on $R^i \cup S^i$ for which $\Phi(\xi, \eta, \nu)$ is the boundary function. Method E then extends immediately by making the following modification. Consider the interior problem defined on $R^i \cup S^i$ for which $\Phi(\xi, \eta, \nu)$ is given on S^i and $v_{\xi\xi} + v_{\eta\eta} + v_{\nu\nu} = 0$ is to be valid on R^i. Approximate v by V by applying the three dimensional generalization, (GREENSPAN [5], Ch. VI) of (2.8) in a fashion completely analogous to that described in Method E. Finally, to approximate u by U apply (3.4) with v, u replaced by V, U, respectively.

The difference then in the methods for E^2 and E^3 lies in the multiplication of V by $\sqrt{\xi^2+\eta^2+\nu^2}$ to obtain U in E^3, a step which was unnecessary in E^2.

All the uniqueness and convergence theorems are again valid for the extended numerical method. However, instead of giving next an example to demonstrate the ease with which the method can often be applied in E^3, we shall couple the ideas above with some further results to show how our method yields directly estimates of electrostatic capacity.

4. - Capacity.

The electrostatic capacity C of a closed surface S is the charge which, in electrostatic equilibrium on S, raises the potential of S to unity. For analytical purposes, then, it is usually convenient to use the formula

$$C = \frac{1}{4\pi} \iint_S \frac{\partial u}{\partial n} \, \mathrm{d}A \,, \tag{4.1}$$

where $\varphi \equiv 1$ on S, u is the solution of the associated exterior DIRICHLET problem for S, and $\frac{\partial u}{\partial n}$ is the outward normal derivative of u on S. Analytical methods, like those associated with isoperimetric inequalities, have bee napplied to yield upper and lower bounds for C for a large number of special surfaces S (see, e. g., PÓLYA and SZEGÖ [3], Ch. III). We shall be concerned, however, with a high speed computer technique for a *direct* estimation of C. For our purpose (4.1) is relatively unwieldy and the following lemma will be of fundamental importance in finding a convenient replacement. The notation is that of Section 3.

Lemma 4.1. *Let R_1 be the interior of S and let the origin be a point in R.*[1] *Let $v(\xi, \eta, \nu)$ and C be defined by* (3.4) *and* (4.1), *respectively. Then,*

$$C = v(0,\ 0,\ 0)\,. \tag{4.2}$$

Proof. It is known (KELLOG [5], p. 144, Theorem V) that

$$C = \lim_{\sqrt{x^2+y^2+z^2} \to \infty} \left\{ u(x,\ y,\ z) \sqrt{x^2+y^2+z^2} \right\}. \tag{4.3}$$

Then, since $x^2+y^2+z^2 = (\xi^2+\eta^2+\nu^2)^{-1}$, it follows from (3.4) and (4.3) that

$$C = \lim_{\sqrt{\xi^2+\eta^2+\nu^2} \to 0} v(\xi,\ \eta,\ \nu)\,. \tag{4.4}$$

Since $v(\xi, \eta, \nu)$ is continuous, (4.2) follows immediately from (4.4) and the lemma is proved.

The significance of (4.2) from the computational point of view is that if we wish to estimate the capacity C of S, we need only invert the associated exterior DIRICHLET problem as shown in Section 3 and approximate the harmonic function $v(\xi, \eta, \nu)$ *at the origin* to yield the result. From the point of view of modern computational machinery, this can be done easily and in a highly efficient manner. To illustrate the power of the method, we shall now approximate the capacity of a unit cube, a classical quantity about which extensive analytical work has yielded only the bounds $0.632 < C < 0.71055$ (PÓLYA and SZEGÖ [3], pp. 76-78).

Let S be the cube whose edges are parallel to the coordinate axes, whose faces are parallel to the base planes, whose center is at (0, 0, 0), and whose edge has unit length. In discussing the capacity of S, symmetry considerations allow us to focus attention only on that triangular portion S_{48} of S (consult Diagram 4.1) whose vertices are $\left[0,\ 0,\ \frac{1}{2}\right]$, $\left[\frac{1}{2},\ 0,\ \frac{1}{2}\right]$, and $\left[\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right]$. Under mapping (3.2), it follows that $\left[0,\ 0,\ \frac{1}{2}\right] \to (0,\ 0,\ 2)$, $\left[\frac{1}{2},\ 0,\ \frac{1}{2}\right] \to (1,\ 0,\ 1)$, $\left[\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right] \to$ $\to \left[\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3}\right]$ and $S_{48} \to S^i_{48}$, where S^i_{48} is the spherical region shown in Diagram 4.1. The problem is there by restricted to the bounded region R^i_{48} whose boundary surfaces are S^i_{48} and the planes π_1, π_2, π_3 whose equations are $\xi = \eta$, $\eta = 0$, $\xi = \nu$. For fixed grid size h, a grid is constructed in R^i_{48} (consult GREENSPAN [4], pp. 84-87). At each resulting interior grid point (ξ, η, ν), as shown in Diagram 4.2, the LAPLACE analogue

$$
(4.5)\quad \left\{
\begin{aligned}
&\left[-\frac{2}{h_1h_2}-\frac{2}{h_3h_4}-\frac{2}{h_5h_6}\right] V(\xi,\ \eta,\ \nu) + \frac{2}{h_1(h_1+h_2)} V(\xi + h_1,\ \eta,\ \nu) +\\
&+ \frac{2}{h_2(h_1+h_2)} V(\xi - h_2,\ \eta,\ \nu) \qquad + \frac{2}{h_3(h_3+h_4)} V(\xi,\ \eta + h_3,\ \nu) +\\
&+ \frac{2}{h_4(h_3+h_4)} V(\xi,\ \eta - h_4,\ \nu) \qquad + \frac{2}{h_5(h_5+h_6)} V(\xi,\ \eta,\ \nu + h_5) +\\
&\qquad\qquad + \frac{2}{h_6(h_5+h_6)} V(\xi,\ \eta,\ \nu - h_6) = 0,
\end{aligned}
\right.
$$

where $(\xi + h_1, \eta, \nu)$, $(\xi - h_2, \eta, \nu)$, $(\xi, \eta + h_3, \nu)$, $(\xi, \eta - h_4, \nu)$, $(\xi, \eta, \nu + h_5)$

and $(\xi, \eta, \nu - h_6)$ are the neighbors of (ξ, η, ν) is applied. At each boundary grid

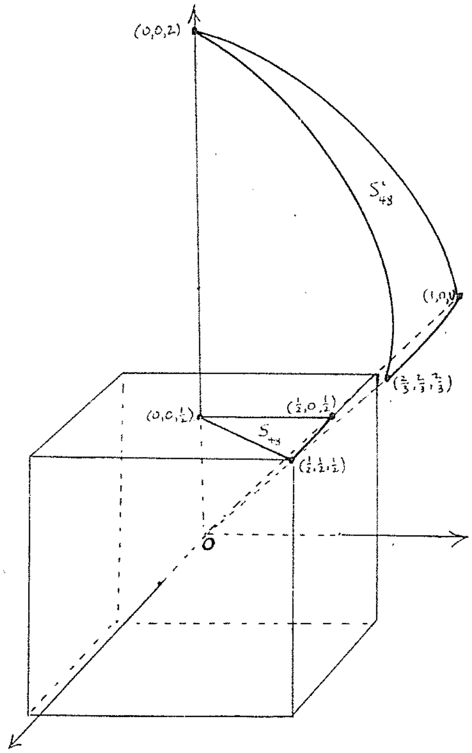

Diagram 4.1

point on π_1, π_2 and π_3, (4.5) is applied with the appropriate symmetry condition. The known boundary values $\Phi = (\xi^2 + \eta^2 + \nu^2)^{-1/2}$ on S^i_{48} are inserted

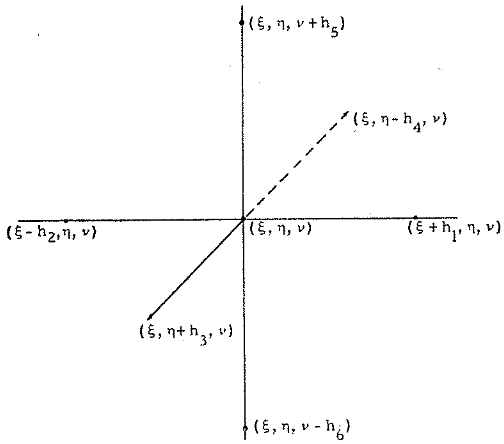

Diagram 4.2

wherever possible. The resulting linear equations are then solved in the spirit of the illustrative example in Section 2. The computer need print out only $V(0, 0, 0) = C$.

The five cases actually computed corresponded to grid sizes $h = .2, .15, .1, .07$ and $.05$. The first four were run on the CDC 1604 and the fifth was run on the CDC 3600. The respective values of C which resulted were 0.66085, 0.66088, 0.66152, 0.66059 and 0.66098. We conclude that a reasonable estimate of C is 0.661.

References.

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Summary.

A digital computer technique is described for approximating the solution of the exterior Dirichlet *problem. The essence of the method is the combination of geometric inversion with a method for approximating the solution of an interior* Dirichlet *problem. Two and three dimensional problems are discussed separately because each requires a different treatment of boundary data. The technique for three dimensions extends to higher dimensional problems. Application is made to the calculation of electrostatic capacity by reducing the usual formula for capacity to one which requires only the evaluation at the origin of the solution of a particularly simple interior* Dirichlet *problem. Finally, the capacity of a unit cube is estimated to be* 0.661.

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