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A Note on Nörlund Means. (**)

1. – Let $\{p_n\}$ be a sequence of complex numbers with

$$P_n = \sum_{\nu=0}^{n} p_\nu \neq 0 \quad (n \geqslant 0) \quad \text{and} \quad P_n = 0 \quad (n < 0) .$$

Then we define the NÖRLUND mean or the (N, p) mean of the sequence $s_n = \sum_{\nu=0}^{n} a_\nu$ of the infinite series of complex numbers $\sum a_n$, by means of the transformation:

$$t_n^p = P_n^{-1} \cdot \sum_{\nu=0}^{n} p_{n-\nu}\, s_\nu .$$

The necessary and sufficient conditions for the regularity of (N, p) method (see [4], Theorem 2) are

(1) $$p_n = o(P_n) \qquad (n \to \infty),$$

and

(2) $$P_n^* = \sum_{\nu=0}^{n} | p_\nu | = O(P_n) .$$

And also the necessary and sufficient conditions for the absolute regularity (see [5]) are (1) and, as $n \to \infty$,

(3) $$\sum_{n=\nu}^{\infty} \left| \frac{P_{n-\nu}}{P_n} - \frac{P_{n-\nu-1}}{P_{n-1}} \right| = O(1) ,$$

uniformly in ν.

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Since (1) is equivalent to

$$P_{n-1} = P_n + o(P_n), \tag{4}$$

with every regular and absolutely regular (N, p) method there are associated power series

$$P(z) = \sum_{n=0}^{\infty} P_n z^n, \qquad p(z) = (1 - z) P(z) = \sum_{n=0}^{\infty} p_n z^n,$$

which are convergent for $|z| < 1$.

Let the sequence $\{q_n\}$ be defined similarly as $\{p_n\}$. We define the sequence of constants $\{c_n\}$ $\{k_n\}$ and $\{l_n\}$ formally by means of the following identities:

$$(5) \quad \begin{cases} p^{-1}(z) = \sum_{n=0}^{\infty} c_n z^n = c(z) & (c_{-1} = 0) \\ q(z)\, p^{-1}(z) = \sum_{n=0}^{\infty} k_n z^n = k(z) & (k_{-1} = 0) \\ p(z)\, q^{-1}(z) = \sum_{n=0}^{\infty} l_n z^n = l(z) & (l_{-1} = 0). \end{cases}$$

If all the series summable by the method A are also summable by the method B, then we say that A implies B and we write A ⊂ B. If A ⊂ B and B ⊂ A, then the two methods are said to be equivalent and we write A ∼ B. If A is equivalent to convergence we say that A is *ineffective*.

2. – The main object of this Note is to obtain a factor (Theorem 1) for absolute Nörlund summability. This problem has its origin in the following problem: In the case of Cesàro summability we know that $|C, \alpha| \subset |C, \beta|$ where $0 \leqslant \alpha \leqslant \beta$, that is, if $\{\sigma_n^\alpha\}$ denotes the Cesàro mean order α of $\{s_n\}$, then $\sum_n |\Delta\sigma_n^\alpha| < \infty$ implies $\sum_n |\Delta\sigma_n^\beta| < \infty$. The converse problem has been tackled by various authors in considering $|C, \alpha|$ summability of the factored series $\sum \varepsilon_n a_n$ whenever $\sum a_n$ is summable $|C, \beta|$. But here our attempt is in a new direction. Instead of affecting the series by multiplying the factor $\{\varepsilon_n\}$, we affect the difference of the mean, that is, given $\sum |\Delta\sigma_n^\beta| < \infty$, we in particular investigate the necessary and sufficient conditions on $\{\varepsilon_n\}$ such that $\sum |\varepsilon_n| \cdot |\Delta\sigma_n^\alpha| < \infty$. We consider this problem in the Nörlund summability set up. To appreciate the importance of Theorem 1, we give below a result which follows as a corollary from Theorem 1.

Theorem 1 (a). *Let* $0 \leqslant \alpha \leqslant \beta$. *Then the necessary and sufficient condition that* $\sum |\varepsilon_n| |\Delta\sigma_n^\alpha| < \infty$ *whenever* $\sum |\Delta\sigma_n^\beta| < \infty$, *is that* $\varepsilon_n = O(n^{\alpha-\beta})$.

In particular, taking $\alpha = 0$, *we obtain* $\varepsilon_n = O(n^{-\beta})$ *to be the necessary and sufficient condition for absolute convergence of* $\sum \varepsilon_n a_n$ *whenever* $\sum a_n$ *is summable* $|C, \beta|$.

Incidentally, it is seen that Theorem 1 is instrumental in obtaining Theorems 2, 3 and 4 which are theorems of inclusion and equivalence of two Nörlund methods. From Theorem 3 and 4, in particular, we obtain different type of theorems for the ineffectiveness of $|N, p|$ method. Lastly Theorem 5 and 6 which give sufficient condition for the ineffectiveness of (N, p) and $|N, p|$ method are deduced from known results.

We now state the following theorems.

Theorem 1. *Let condition* (2) *and the following hold*:

(6) $$k(z) = \sum_{n=0}^{\infty} k_n z^n \quad \textit{is convergent in} \quad |z| \leqslant 1,$$

(7) $$|P_\nu| \sum_{n=\nu}^{\infty} \frac{|q_n|}{|P_n Q_{n-1}|} = O(1).$$

Then the necessary and sufficient condition that $\sum |\varepsilon_n| |\Delta t_n^q| < \infty$, *whenever* $\sum |\Delta t_n^p| < \infty$, *is that* $\varepsilon_n = O(|Q_n| / |P_n|)$.

Theorem 2. *Let condition* (2) *and* (6) *hold. Then the necessary and sufficient condition that* $|N, p| \subset |N, q|$ *is that*

(8) $$P_n = O(Q_n).$$

Theorem 3. *Let* (2) *and* (6) *and the following hold:*

(9) $$l(z) = \sum_{n=0}^{\infty} l_n z^n \quad \textit{is convergent in} \quad |z| \leqslant 1.$$

Then the necessary and sufficient condition that the $|N, p|$ *is equivalent to* $|N, q|$ *is that* (8) *should hold.*

Theorem 4. *Let* $p(z)$ *and* $q(z)$ *be regular in* $|z| < 1$. *Let* (2), (6), (9) *and the following hold:*

(10) $$Q_n^* = \sum_{\nu=0}^{\infty} |q_\nu| = O(Q_n).$$

Then $|N, p| \sim |N, q|$.

Theorem 5. *Let* $p_n \geqslant 0$. *If*

(11) $$\lim P_n < 2p_0,$$

then (N, p) *is ineffective. The inequality* (11) *is best possible in the sense that it can not be improved.*

Theorem 6. *Let* $p_n \geqslant 0$ *and non-increasing such that* (11) *holds. Then* $|N, p|$ *is ineffective and in this case also* (11) *is best possible.*

It may be pointed out here that recently MISENGER [6] has proved, in particular, the following

Theorem. *Let* $|N, p|$, $|N, q|$ *be two absolutely regular method such that*

$$\sum_n |k_n| < \infty, \qquad \sum_n |l_n| < \infty.$$

Then $|N, p| \sim |N, q|$.

Comparing this theorem of MISENGER with Theorem 3 and 4 of ours we note the following facts.

Instead of assuming absolute regularity of the methods we have assumed (2) (in Theorem 3) which is one of the regularity condition of (N, p) method. Though (1) is common to both the set of conditions of regularity and absolute regularity of (N, p), yet we know that, in general, regularity and absolute regularity are independent notions. Of course, in case of (N, p) method, *if* P_n *is bounded, regularity implies absolute regularity*; for (2), implies that

(a) $$|P_n| \geqslant K^{-1} \cdot |p_0|,$$

for some positive constant K (which is not necessarily the same at each occurrence) and (2), with the boundedness of $\{P_n\}$ implies

(b) $$P_n \in \mathrm{BV}.$$

Also it follows from (b) that $p_n = o(1)$; and from (a) and (b) that $P_n \to P \ (\neq 0)$. Now it is easy to see that $p_n = o(P_n)$, and $\{P_{n-\nu}/P_n\}$ is of bounded variation in n uniformly in ν so that $|N, p|$ is absolutely regular.

3. – We require the following lemma for the proof of Theorem 1.

Lemma 1. *The necessary and the sufficient condition that* $\{t_n^p\} \in \mathrm{BV}$ *implies* $\sum |\varepsilon_n|\,|\Delta t_n^q| < \infty$, *is that*

$$\sum_{n=\varrho+1}^{\infty} |S_{n,\varrho}| = O(1)\,, \tag{12}$$

uniformly in ϱ, *where*

$$S_{n,\varrho} = \begin{cases} \varepsilon_n \sum\limits_{\nu=0}^{\varrho} G(n,\,\nu) & (\varrho \leqslant n) \\ 0 & (\varrho > n)\,; \end{cases}$$

and

$$G(n,\,\nu) = (k_{n-\nu}/Q_n - k_{n-\nu-1}/Q_{n-1})\,P_\nu\,.$$

Proof. We have, from (5),

$$Q_n = \sum_{\nu=0}^{n} k_{n-\nu}\,P_\nu\,, \tag{13}$$

and so

$$\sum_{\nu=0}^{n} G(n\,,\,\nu) = 0\,. \tag{14}$$

Using (14) and the familiar inversion formula

$$t_n^q = Q_n^{-1} \sum_{\nu=0}^{n} k_{n-\nu}\,P_\nu\,t_\nu^p\,,$$

we have

$$\varepsilon_n\,\Delta t_n^q = \varepsilon_n \sum_{\nu=0}^{n} G(n,\,\nu)\,t_\nu^p = \sum_{\nu=0}^{n} \Delta t_\nu^p\,S_{n,\nu}\,. \tag{15}$$

Now the lemma follows from the transformation (5) by using the theorem of Knopp and Lorentz [5].

Proof of Theorem 1. *Sufficiency.* It is enough to show that the condition of Lemma 1 is satisfied. Since $\varepsilon_n = O(Q_n/P_n)$, it is enough to prove that

$$J_\varrho = \sum_{n=\varrho+1}^{\infty} \left| \frac{Q_n}{P_n} \sum_{\nu=0}^{\varrho} G(n, \nu) \right| = O(1) ,$$

uniformly in ϱ. Now since

$$\sum_{\nu=0}^{\varrho} (k_{n-\nu} - k_{n-\nu-1}) P_\nu = \sum_{\nu=0}^{\varrho} p_\nu k_{n-\nu} - P_\varrho \; k_{n-\varrho-1} ,$$

we have

$$J_\varrho \leqslant \sum_{n=\varrho+1}^{\infty} \left| \frac{1}{P_n} \sum_{\nu=0}^{\varrho} (k_{n-\nu} - k_{n-\nu-1}) P_\nu \right| + \sum_{n=\varrho+1}^{\infty} \left| \frac{q_n P_n}{Q_{n-1}} \sum_{\nu=0}^{\varrho} k_{n-\nu-1} \; P_\nu \right|$$

$$\leqslant \sum_{n=\varrho+1}^{\infty} \frac{1}{|P_n|} \sum_{\nu=0}^{\varrho} |p_\nu| \, |k_{n-\nu}| + |P_\varrho| \sum_{n=\varrho+1}^{\infty} \frac{|k_{n-\varrho-1}|}{|P_n|} +$$

$$+ \sum_{n=\varrho+1}^{\infty} \frac{|q_n|}{|P_n Q_{n-1}|} \sum_{\nu=0}^{\varrho} |k_{n-\nu-1}| \, |P_\nu| = J_\varrho^{(1)} + J_\varrho^{(2)} + J_\varrho^{(3)}, \qquad \text{say.}$$

Now by (2), for $m \geqslant n$,

$$|P_n| \leqslant P_n^* \leqslant P_m^* = O(P_m) .$$

Hence, by hypotheses,

$$J_\varrho^{(1)} = \sum_{\nu=0}^{\varrho} |p_\nu| \sum_{n=\varrho+1}^{\infty} \frac{|k_{n-\nu}|}{|P_n|} \leqslant K \frac{1}{|P_\nu|} \sum_{\nu=0}^{\varrho} |p_\nu| \sum_{n=\varrho+1}^{\infty} |k_{n-\nu}|$$

$$\leqslant K \frac{1}{|P_\nu|} \sum_{\nu=0}^{\varrho} |p_\nu| \leqslant K ,$$

$$J_\varrho^{(2)} \leqslant \frac{|P_\varrho|}{|P_\varrho^*|} \sum_{n=\varrho+1}^{\infty} |k_{n-\varrho-1}| \leqslant K ,$$

and lastly also:

$$J_\varrho^{(3)} \leqslant K |P_\varrho| \sum_{n=\varrho+1}^{\infty} \frac{|q_n|}{|P_n Q_{n-1}|} \sum_{\nu=0}^{\varrho} |k_{n-\nu-1}| \leqslant K |P_\varrho| \sum_{n=\varrho+1}^{\infty} \frac{|q_n|}{|P_n Q_{n-1}|} \leqslant K .$$

Necessity. It follows from Lemma 1 that the condition $S_{\varrho+1,\varrho+1} = O(1)$ is necessary. But

$$S_{\varrho+1,\varrho+1} = \varepsilon_{\varrho+1}\left(\frac{k_0}{Q_{\varrho+1}} - \frac{k_{-1}}{Q_\varrho}\right)P_{\varrho+1} = \varepsilon_{\varrho+1}\, P_{\varrho+1}/Q_{\varrho+1}\,,$$

whence follows the necessity.

We now obtain the

Corollary 1. *The necessary and sufficient condition that* $\{\sigma_n^\alpha\} \in \mathrm{BV}$, $\alpha > 0$, *implies* $\sum |\varepsilon_n|\,|\Delta t_n| < \infty$, *where* $\{t_n\}$ *is the* $\left(\mathrm{N}, \dfrac{1}{n+1}\right)$ *mean of* $\{s_n\}$, *is that* $\varepsilon_n = O\big((\log n)/n^\alpha\big)$.

Proof. When $q_n = \dfrac{1}{n+1}$ and $p_n = A_n^{\alpha-1}$, then it has been proved elsewhere that [3]

$$k(z) = \left\{\log \frac{1}{1-z}\right\}(1-z)^\alpha \qquad (\alpha > 0)\,,$$

con be exchanded in a power series which is convergent in $|z| \leqslant 1$. It is easy to see that other conditions of the lemma are also satisfied.

Corollary 2 (see [2]). *Let* (2) *hold. Also let* $\sum |c_n| < \infty$. *Then the necessary and sufficient condition that* $\sum \varepsilon_n a_n$ *should be absolutely convergent whenever* $\sum a_n$ *is summable* $|\mathrm{N}, p|$ *is that* $\varepsilon_n P_n = O(1)$.

Proof. Put $q_0 = 1$, $q_n = 0$ $(n \geqslant 1)$.

4. – We require the following lemmas for the proof of Theorem 2.

Lemma 2. *The necessary and sufficient condition that* $|\mathrm{N}, p| \subset |\mathrm{N}, q|$ (*that is* $t_n^p \in \mathrm{BV}$, $t_n^p \to s$ *imply* $t_n^q \in \mathrm{BV}$, $t_n^q \to s$) *are that*

$$k_{n-\nu} = o(Q_n) \qquad (\nu \text{ fixed}) \tag{16}$$

and

$$\sum_{n=\varrho+1}^{\infty} \Big| \sum_{\nu=0}^{\varrho} G(n, \nu) \Big| = O(1)\,, \tag{17}$$

uniformly in $\varrho \geqslant 0$.

Considering the transformation (15) with $\varepsilon_n = 1$, the lemma follows by appealing to the theorem of KNOPP and LORENTZ [5].

Lemma 3. *Conditions* (2), (6) *and* (8) *togather imply* (10) *and also* (7).

Proof. Since by (5)

$$q_n = \sum_{\varrho=0}^{n} k_{n-\varrho}\, p_\varrho ,$$

we have

$$Q_n^* \leqslant \sum_{\nu=0}^{n} \sum_{\varrho=0}^{\nu} | k_{\nu-\varrho} | \, | p_\varrho | = \sum_{\varrho=0}^{n} | p_\varrho | \sum_{\nu=\varrho}^{n} | k_{\nu-\varrho} | \leqslant K \sum_{\varrho=0}^{n} | p_\varrho | = O(P_n) = O(Q_n) ,$$

and this proves (10). Next writing $| q_n | = Q_n^* - Q_{n-1}^*$, and noting that

$$Q_n = O(P_n) \tag{18}$$

[this follows from relation (13) by using (2) and (6)], we observe that the condition (7) is satisfied if

$$| P_\nu | \sum_{n=\nu}^{\infty} \frac{Q_n^* - Q_{n-1}^*}{| Q_n Q_{n-1} |} = O(1) . \tag{19}$$

Now by (10) we have

$$Q_n^* \leqslant Q_m^* \leqslant K \, | Q_m | \qquad (m \geqslant n) ,$$

and so left hand side of (19) is equal to

$$O\left(| P_\nu | \sum_{n=\nu}^{\infty} \frac{Q_n^* - Q_{n-1}^*}{| Q_n^* Q_{n-1}^* |} \right)$$

or $O(|P_\nu| \,/\, Q_\nu^*)$ which is itself $O(1)$ because of (10) and (8).

Proof of Theorem 2. Put $\varepsilon_n = 1$ in Theorem 1 and we get the necessary and sufficient condition so that $t_n^p \in \mathrm{BV}$ implies $t_n^q \in \mathrm{BV}$. Now because of Theorem 1, Lemma 2 [note that the condition (17) is the same as the condition of Lemma 1 with $\varepsilon_n = 1$] and Lemma 3, we have only to prove (16).

Now since by (6) $\sum | k_n | < \infty$, we have $k_n = o(1)$. Also by (10), $|Q_n|^{-1} < K \, | q_0 |^{-1}$. Hence, uniformly in ν,

$$\left| \frac{k_{n-\nu}}{Q_n} \right| \leqslant K \, | q_0 |^{-1} \; | k_{n-\nu} | = o(1)$$

as $n \to \infty$, and this proves the theorem.

Proof of Theorem 3. Considering the inclusion $|N, p| \subset |N, q|$ and $|N, q| \subset |N, p|$ in the light of Theorem 2 and taking into account Lemma 3 and relation (18), we immediately see the truth of the sufficiency part of the theorem. Since $|N, p| \subset |N, q|$ is necessary when they are equivalent, the necessity of condition (8) follows from Theorem 2.

Proof of Theorem 4. When (2), (6) and (9) hold, then (8) is equivalent to (10); for, since by (5) $P_n = \sum_{\nu=0}^{n} l_{n-\nu} Q_\nu$, we obtain (8). Now the other implication is contained in Lemma 3. Thus it follows from Theorem 3 that (2), (6), (9) and (10) are sufficient conditions for equivalence.

5. – To prove Theorem 5 and 6, we require the following lemmas.

Lemma 4 (see [1]). *Let A be a regular triangular sequence-to-sequence transformation given by $t_n = \sum_{\nu=0}^{n} \alpha_{n,\nu} s_\nu$. Suppose that*

$$\lim\inf \left\{ |\alpha_{n,n}| - \sum_{\nu=0}^{n-1} |\alpha_{n,\nu}| \right\} > 0.$$

Then the method A is ineffective.

Lemma 5 (see [6]). *Let A be an absolutely regular semilower matrix $[\alpha_{n\nu}]$ such that*

$$\lim\inf \left\{ |\alpha_{n,n}| - \sum_{k=n+1}^{\infty} \left| \sum_{i=n}^{k} (\alpha_{k,i} - \alpha_{k-1,i}) \right| \right\} > 0.$$

Then $|A|$ is ineffective.

Proof of Theorem 5. We note that $\alpha_{n,\nu} \geqslant 0$ for all n and ν; and so the condition of the Lemma 4 takes a simpler form. It is easily seen that in our case it reduces to

$$\lim\inf \frac{p_0}{P_n} > \frac{1}{2},$$

and this inequality can not be improved, for consider the case

$$p_0 = 1, \qquad p_1 = c > 0, \qquad p_n = 0 \ (n \geqslant 2). \tag{20}$$

It is clear that (N, p) sums the sequence $(-c)^n$ to the value 0 but it diverges if $c \geqslant 1$.

P r o o f o f T h e o r e m 6. By Lemma 5, we have only to show that

$$(21)\qquad \liminf \left\{\frac{p_0}{P_n} - \sum_{k=n+1}^{\infty} \left| \frac{P_{k-n}}{P_k} - \frac{P_{k-n-1}}{P_{k-1}} \right| \right\} > 0 .$$

Now by hypotheses, and for $n = 0, 1, \ldots, k-1$,

$$p_{k-\nu}/P_k \leqslant p_{k-\nu-1}/P_{k-1} ,$$

and so, for $n > k \geqslant 1$,

$$\frac{P_{n-k}}{P_n} = 1 - \sum_{\nu=0}^{k-1} \frac{p_{n-\nu}}{P_n} \geqslant 1 - \sum_{\nu=0}^{\varrho-1} \frac{p_{n-1-\nu}}{P_{n-1}} = \frac{P_{n-k-1}}{P_{n-1}} .$$

Therefore, noting that when p_n is non-negative and non-increasing (4) is true, we have

$$\sum_{k=n+1}^{\infty} \left| \frac{P_{k-n}}{P_k} - \frac{P_{k-n-1}}{P_{k-1}} \right| = \sum_{k=n+1}^{\infty} \left(\frac{P_{k-n}}{P_k} - \frac{P_{k-n-1}}{P_{k-1}} \right) = 1 - \frac{p_0}{P_n} ,$$

so that (21) reduces to (11).

Considering the example (20) we observe that $(-c)^n$ is summable $|\mathrm{N}, p|$ to 0, but it is not of bounded variation when $c \geqslant 1$.

C o r o l l a r y. *Methods* $\left(\mathrm{N}, \frac{1}{(n+1)^2}\right)$ *and* $\left|\mathrm{N}, \frac{1}{(n+1)^2}\right|$ *are ineffective.*

We only note that

$$\lim_{n\to\infty} P_n = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} < 2 .$$

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