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WILLIAM D. L. APPLING (*)

Summability of Real-Valued Set Functions. (**)

1. - Introduction.

Suppose U is a set, F is a field of subsets of U, R is the set of all real-valued functions defined on F, and m is a nonnegative-valued, finitely additive element of R.

In this paper we define a set W^* (see below) of « m-summable » elements of R and develop basic properties of the « m-summability operator » s (see below) that we define on W^*, proving, among other things, m-summability analogues of some theorems of R. HENSTOCK.

Suppose:

1) R^+ is the set of all nonnegative-valued elements of R.

2) R_A is the set of all finitely additive elements of R.

3) $R_A^+ = R^+ \cap R_A$.

4) C is the set of all elements of R_A absolutely continuous with respect to m.

5) W is the set to which H belongs if and only if:

a) H is in R^+.

b) For each number $K \geqslant 0$, the integral (section 2)

$$\int_U \min\{K,\ H(V)\}\, m(V)$$

(*) Indirizzo: Department of Mathematics, North Texas State University, Denton, Texas, U.S.A. .

(**) Ricevuto: 4-IX-1965.

exists, and

$$\text{c)} \qquad \infty > \sup \int_U \min\{K,\ H(V)\}\ m(V), \qquad \text{for} \qquad 0 \leqslant K.$$

6) For each H in W, $s(H)$ is the function on F such that for each V in F,

$$s(H)(V) = \sup \int_V \min\{K,\ H(I)\}\ m(I)\,, \qquad \text{for} \qquad 0 \leqslant K.$$

7) W^* is the set of all H in R of the form $Y - Y'$, for Y and Y' in W.

We prove (Theorem 6.1) that *if H is in W and P is in R^+, then $H + P$ is in W if and only if P is in W, in which case*

$$s(H + P) = s(H) + s(P)\,.$$

Theorem 6.1 immediately implies (Lemma 12.1) that *if each of H', H^*, P' and P^* is in W and $H' - P' = H^* - P^*$, then $s(H') - s(P') = s(H^*) - s(P^*)$.*

We show (Theorem 12.1) that $W = W^* \cap R^+$, *and if each of H, P and P' is in W and $H = P - P'$, then $s(H) = s(P) - s(P')$.*

The consistency implied by Lemma 12.1 and Theorem 12.1 makes possible the following

Definition: *If H is in W^*, then $s(H)$ is the function x, on F, such that if each of Y and Y' is in W and $H = Y - Y'$, then $x = s(Y) - s(Y')$.*

We demonstrate (Theorem 12.5) that if each of H and P is in W^* and H is bounded, then $H\,P$ is in W^* and, for each V in F,

$$s(H\,P)(V) = \int_V H(I)\ s(P)(I)\,.$$

We have (Theorem 12.4), as an analogue of a theorem of Henstock [5], that if each of H and P is in W^*, then each of $\max\{H,\ P\}$ and $\min\{H,\ P\}$ is in W^* and, for each V in F,

$$s(\max\{H,\ P\})(V) = \int_V \max\{s(H)(I),\ s(P)(I)\}$$

and

$$s(\min\{H,\ P\})(V) = \int_V \min\{s(H)(I),\ s(P)(I)\}.$$

We show (Theorem 9.1) another analogue of a theorem of Henstock [5], demonstrating that if each of H and P is in W and $0 < q < 1$, then $H^q\,P^{1-q}$

is in W and, for each V in F,

$$s(H^q P^{1-q})(V) = \int_V s(H)(I)^q \; s(P)(I)^{1-q}.$$

We prove (Theorem 10.1) that *if H is in W and $1 < q$, then H^q is in W if and only if $\int_U s(H)(V)^q m(V)^{1-q}$ exists, is which case, for each V in F,*

$$s(H^q)(V) = \int_V s(H)(I)^q \; m(I)^{1-q}.$$

Finally, we show (Theorem 12.6) that *if H is in W^*, and for some number $T > 0$, $|H| - T$ is in R^+, then*

$$\int_U m(V)/H(V)$$

exists.

2. - Preliminary theorems and definitions.

If V is in F, then the statement that D is a subdivision of V means that D is a finite collection of mutually exclusive sets of F whose union is V.

If D is a subdivision of a set V of F, then the statement that E is a refinement of D means that E is a subdivision of V, every set of which is a subset of some set of D.

Throughout this paper all integrals discussed will be HELLINGER [4] type limits (i. e., for refinements of subdivisions) of the appropriate sums. Thus, if P is in R, then $\int_U P(V)$ exists if and only if for each V in F, $\int_V P(I)$ exists, in which case the function M on F defined by $M(V) = \int_V P(I)$ is in R_A.

Suppose P is in R.

Suppose V is in F and T is a number such that if D is a subdivision of V, then $|\sum_D P(I)| \leqslant T$. This implies that if V' is in F and $V' \subseteq V$, and $L(V')$ and $G(V')$ are the respective sup and inf of the set of all sums $\sum_{D'} P(I)$, where D' is a subdivision of V', then $-\infty < G(V') \leqslant L(V') < \infty$. We see that if E is a refinement of each of the subdivisions D and D^* of V, then $\sum_D G(I) \leqslant \leqslant \sum_E G(I) \leqslant \sum_E P(I) \leqslant \sum_E L(I) \leqslant \sum_{D^*} L(I)$, so that each of $\int_V G(I)$ and $\int_V L(I)$ exists, $\int_V G(I) \leqslant \int_V L(I)$, and $\int_V P(I)$ exists if and only if $\int_V G(I) = \int_V L(I)$, in which case $\int_V G(I) = \int_V P(I) = \int_V L(I)$.

We state without proof a theorem of KOLMOGOROFF [6].

Theorem 2. K. *If $\int_U P(V)$ exists, then*

$$\int_U \big|\, |P(V)| - |\textstyle\int_V P(I)| \,\big| = \int_U |P(V) - \textstyle\int_V P(I)| = 0,$$

so that if V is in F, then

$$\int_V |P(I)|$$

exists if and only if

$$\int_V \left| \int_I (J) \right|$$

exists, in which case equality holds.

We state an immediate consequence of Theorem 2.K.

Corollary 2.K. *If H is in R and bounded and* $\int_U P(V)$ *exists, then*

$$\int_U |H(V)|\,|P(V) - \int_V P(I)| = 0 ,$$

so that if V is in F, then

$$\int_V H(I)\, P(I)$$

exists if and only if

$$\int_V H(I) \int_I P(J)$$

exists, in which case equality holds.

Suppose each of a, b, c and d is a number.
We state some inequalities:

$$\min\{a+b,\ c+d\} \geqslant \min\{a,\ c\} + \min\{b,\ d\}. \tag{2.1}$$

$$\max\{a+b,\ c+d\} \leqslant \max\{a,\ c\} + \max\{b,\ d\}. \tag{2.2}$$

$$a^q c^{1-q} + b^q d^{1-q} \leqslant (a+b)^q (c+d)^{1-q} , \tag{2.3}$$

for $0 < q < 1$ and $0 \leqslant \min\{a,\ b,\ c,\ d\}$.

We adopt the convention that if each of y and z is a number, then $y/z = 0$ if $z = 0$, and has the usual meaning otherwise.

$$a^q c^{1-q} + b^q d^{1-q} \geqslant (a+b)^q (c+d)^{1-q} , \tag{2.4}$$

for $1 < q$ and $0 \leqslant \min\{a,\ b,\ c,\ d\}$, provided that $a = 0$ if $c = 0$, and

$b = 0$ if $d = 0$.

(2.5) $$\min\{a,\ b\} - \min\{a,\ c\} \leqslant \min\{a,\ b - c\},$$

if $0 \leqslant \min\{a,\ c,\ b - c\}$.

(2.6) $$|\min\{a,\ b\} - \min\{c,\ d\}| \leqslant |a - c| + |b - d|.$$

(2.7) $$|\max\{a,\ b\} - \max\{c,\ d\}| \leqslant |a - c| + |b - d|.$$

(2.8) $$|a^q b^{1-q} - c^q d^{1-q}| \leqslant a^q |b - d|^{1-q} + |a - c|^q d^{1-q},$$

if $0 \leqslant \min\{a,\ b,\ c,\ d\}$ and $0 < q < 1$.

Now suppose that each of g and h is in R_A and is bounded.
Suppose E is a refinement of the subdivision D of the set V of F.
(2.1) and (2.2) imply that

$$-\int_V (|g(I)| + |h(I)|) \leqslant \sum_E \min\{g(I),\ h(I)\} \leqslant \sum_D \min\{g(I),\ h(I)\} \leqslant$$

$$\leqslant \sum_D \max\{g(I),\ h(I)\} \leqslant \sum_E \max\{g(I),\ h(I)\} \leqslant \int_V (|g(I)| + |h(I)|),$$

so that each of

$$\int_V \min\{g(I),\ h(I)\} \quad \text{and} \quad \int_V \max\{g(I),\ h(I)\}$$

exists.

Suppose each of g and h is in R_A^+.
If $0 < q < 1$, then, by (2.3),

$$0 \leqslant \sum_E g(I)^q\ h(I)^{1-q} \leqslant \sum_D g(I)^q\ h(I)^{1-q},$$

so that

$$\int_V g(I)^q\ h(I)^{1-q}$$

exists.

If $1 < q$ and, for each I in F such that $I \subseteq V$, $g(I) = 0$ if $h(I) = 0$, then (2.4) implies that

$$\sum_D g(I)^q\ h(I)^{1-q} \leqslant \sum_E g(I)^q\ h(I)^{1-q},$$

so that

$$\int_V g(I)^q \ h(I)^{1-q},$$

exists if and only if for some number T and every subdivision E' of V,

$$\sum_{E'} g(I)^q \ h(I)^{1-q} \leqslant T,$$

in which case $\int_V g(I)^q \ h(I)^{1-q}$ is the sup of the set of all such sums.

We see that, together with Theorem 2.K., (2.6), (2.7) and (2.8) (with (2.3)) imply respectively the following theorems, which we state without proof.

Suppose each of H and P is in R and each of $\int_U H(I)$ and $\int_U P(I)$ exists.

Theorem 2.1.

$$\int_U |\min\{H(V),\ P(V)\} - \min\{\int_V H(I),\ \int_V P(I)\}| =$$

$$= \int_U |\max\{H(V),\ P(V)\} - \max\{\int_V H(I),\ \int_V P(I)\}| = 0,$$

so that if V is in F, then

$$\int_V \min\{H(I),\ P(I)\}$$

exists if and only if

$$\int_V \min\{\int_I H(J),\ \int_I P(J)\}$$

exists, in which case equality holds.

A similar statement holds for

$$\int_V \max\{H(I),\ P(I)\} \quad \text{and} \quad \int_V \max\{\int_I H(J),\ \int_I P(J)\}.$$

Theorem 2.2. *If each of H and P is in R^+, and $0 < q < 1$, then*

$$\int_U | H(V)^q \ P(V)^{1-q} - [\int_V H(I)]^q \ [\int_V P(I)]^{1-q} | = 0,$$

so that if V is in F, then

$$\int_V H(I)^q \ P(I)^{1-q}$$

exists and is

$$\int_V [\int_I H(J)]^q \ [\int_I P(J)]^{1-q}.$$

Throughout this paper we shall often simply write integrals whose existence or equivalence to integrals is a simple consequence of one or more of the preceding theorems, and leave the proof of existence or equivalence to the reader.

Theorem 2.3. *If* $0 < q < 1$, *each of* g *and* h *is an element of* R_A^+, *and each of* $\{g_k\}_{k=1}^{\infty}$ *and* $\{h_k\}_{k=1}^{\infty}$ *is a sequence of elements of* R_A^+ *such that*

$$\int_U (|g_n(V) - g(V)| + |h_n(V) - h(V)|) \to 0$$

as $n \to \infty$, *then*

$$\int_U | g_n(V)^q \; h_{n'}(V)^{1-q} - g(V)^q \; h(V)^{1-q} | \to 0$$

as $\min\{n, n'\} \to \infty$.

We now state two equalities that we will need later:

$$a = \max\{a, 0\} + \min\{a, 0\}, \tag{2.9}$$

$$\max\{\min\{a, b\}, \min\{a, c\}\} = \min\{a, \max\{b, c\}\}. \tag{2.10}$$

We end this section by adopting the following notational convention: If an expression is to be written more than once in computations in a given argument, and is of sufficient complexity, it will be enclosed, when it first appears, in brackets with a subscript affixed and will be displayed; thereafter in the argument only the brackets with subscript need be written, e. g.,

$$[X]_r = [\;]_r .$$

3. - Two theorems involving the integral $\int_V g(I)^q \; h(I)^{1-q}$ for g and h in R_A^+ and $1 < q$.

Theorem 3.1. *Suppose* $1 < q$, *each of* f *and* h *is in* R_A^+, *and* $\{g_k\}_{k=1}^{\infty}$ *is a sequence of elements of* R_A^+ *such that:*

1) *If* V *is in* F *and* n *is a positive integer, then* $g_n(V) = 0$ *if* $h(V) = 0$, *and* $g_n(V) \leqslant g_{n+1}(V) \to f(V)$ *as* $n \to \infty$, *and*

2) *For some number* T *and every positive integer* n, $\int_U g_n(V)^q \, h(V)^{1-q} \leqslant T$. *Then:* $\int_U f(V)^q \; h(V)^{1-q}$ *exists, and, for each* V *in* F,

$$\int_V g_n(I)^q \; h(I)^{1-q} \to \int_V f(I)^q \; h(I)^{1-q} \qquad as \qquad n \to \infty .$$

Proof. We see that if V is in F, then $f(V)=0$ if $h(V)=0$.

We first show that $\int_U f(V)^q h(V)^{1-q}$ exists. Suppose D is a subdivision of U. There is a positive integer n such that $0 \leqslant \sum_D f(V)^q h(V)^{1-q} - \sum_D g_n(V)^q h(V)^{1-q} < 1$, so that $\sum_D f(V)^q h(V)^{1-q} < 1 + \sum_D g_n(V)^q h(V)^{1-q} \leqslant 1 + \int_U g_n(V)^q h(V)^{1-q} \leqslant 1 + T$, so that $\int_U f(V)^q h(V)^{1-q}$ exists.

Now suppose $0 < c$ and V is in F. There is a subdivision D of V such that $0 \leqslant \int_V f(I)^q h(I)^{1-q} - \sum_D f(I)^q h(I)^{1-q} < c/2$. There is a positive integer N such that if n is a positive integer $\geqslant N$, then $0 \leqslant \sum_D f(I)^q h(I)^{1-q} - \sum_D g_n(I)^q h(I)^{1-q} < c/2$, so that $0 \leqslant \int_V f(I)^q h(I)^{1-q} - \int_V g_n(I)^q h(I)^{1-q} \leqslant \int_V f(I)^q h(I)^{1-q} - \sum_D g_n(I)^q h(I)^{1-q} < c$. Therefore $\int_V g_n(I)^q h(I)^{1-q} \to \int_V f(I)^q h(I)^{1-q}$ as $n \to \infty$.

Theorem 3.2. *Suppose each of f and h is in R_A^+ and $\{g_k\}_{k=1}^{\infty}$ is a sequence of elements of R_A^+ such that:*

1) *If V is in F and n is a positive integer, then $g_n(V) \leqslant g_{n+1}(V) \to f(V)$ as $n \to \infty$.*

2) *If n is a positive integer, then $\int_U h(V)^2/g_n(V)$ exists, and*

3) *For some number $T > 0$ and all positive integers n, $g_n - T\,h$ is in R^+.*

Then: $\int_U h(V)^2/f(V)$ exists, and, for each V in F,

$$\int_V h(I)^2/g_n(I) \to \int_V h(I)^2/f(I) \qquad as \quad n \to \infty.$$

Proof. Since, for each subdivision D of U, $\sum_D h(V)^2/f(V) \leqslant \sum_D h(V)^2/g_1(V)$, it follows that $\int_U h(V)^2/f(V)$ exists.

We see that $f - Th$ is in R^+.

Now suppose $c > 0$ and V is in F. There is a positive integer N such that if n is a positive integer $\geqslant N$, then $0 \leqslant f(V) - g_n(V) < T^2 c/2$, so that for each subdivision E of V,

$$0 \leqslant \sum_E h(I)^2/g_n(I) - \sum_E h(I)^2/f(I) =$$

$$= \sum_E [h(I)^2/g_n(I)\, f(I)][f(I) - g_n(I)] \leqslant$$

$$\leqslant \sum_E T^{-2}[f(I) - g_n(I)] = T^{-2}[f(V) - g_n(V)] < c/2\,,$$

so that $0 \leqslant \int_V h(I)^2/g_n(I) - \int_V h(I)^2/f(I) \leqslant c/2$. Therefore $\int_V h(I)^2/g_n(I) \to \int_V h(I)^2/f(I)$ as $n \to \infty$.

4. - The integral $\int_V Y(I)\, g(I)$ for g in R_A^+ and Y in R and bounded.

We begin this section by stating some extensions of previous interval function theorems of the author [1], [2]. The proofs carry over for the appropriate elements of R with only slight modifications.

Suppose each of H and P is in R and is bounded, g is in R_A^+, and each of $\int_U H(V)\, g(V)$ and $\int_U P(V)\, g(V)$ exists.

Theorem 4.1. *$\int_U H(V)\, P(V)\, g(V)$ exists.*

Theorem 4.2. *If H is in R^+ and $0 < t$, then $\int_U H(V)^t\, g(V)$ exists.*

Theorem 4.3. *If for some number $T > 0$, $H - T$ is in R^+, then for each V in F, $\int_V g(I)/H(I)$ exists and is $\int_V g(I)^2 / \int_I H(J)\, g(J)$.*

We end this section by stating a theorem which we shall use subsequently.

Theorem 4.4. *If h is a bounded element of R_A, $\{h_k\}_{k=1}^{\infty}$ is a sequence of bounded elements of R_A, and L is a bounded element of R such that for each positive integer n, $\int_U L(V)\, h_n(V)$ exists and $\int_U |h_n(V) - h(V)| \to 0$ as $n \to \infty$, then $\int_U L(V)\, h(V)$ exists, and, for each V in F, $\int_V L(I)\, h_n(I) \to \int_V L(I)\, h(I)$ as $n \to \infty$.*

5. - Some absolute continuity and convergence theorems.

Suppose Q is in W.

We state the following lemma without proof:

Lemma 5.1. *$s(Q)$ is in R_A^+ and*

$$\int_U \left| s(Q)(V) - \int_V \min\{K,\ Q(I)\}\, m(I) \right| \to 0 \quad \text{as} \quad K \to \infty,$$

so that $s(Q)$ is in C.

We state a previous theorem of the author [3].

Theorem 5.A. *If h is in R_A^+, then the following two statements are equivalent:*

1) *h is in C.*

2) *If, for each V in F,*

$$g(V) = \sup \int_V \min\{K\, m(I),\ h(I)\}$$

for $0 \leqslant K$, then g is h.

Theorem 5.1. *If* $0 \leqslant K$ *and* V *is in* F, *then*

$$\int_V \min\{K,\ Q(I)\}\, m(I) = \int_V \min\{K\, m(I),\ s(Q)(I)\}.$$

Proof. If $K \leqslant K'$, then $\int_V \min\{K,\ Q(I)\}\ m(I) = \int_V \min\{K, \min\{K',\ Q(I)\}\}\ m(I) = \int_V \min\{K\, m(I),\ \int_I \min\{K',\ Q(J)\}\ m(J)\}\}$, and $0 \leqslant \int_V \min\{K\, m(I),\ s(Q)(I)\} - \int_V \min\{K\, m(I),\ \int_I \min\{K',\ Q(J)\}\ m(J)\} \leqslant \leqslant \int_V |\, s(Q)(I) - \int_I \min\{K',\ Q(J)\}\ m(J)\,| \to 0$ as $K' \to \infty$. Therefore $\int_V \min\{K,\ Q(I)\}\, m(I) = \int_V \min\{K\, m(I),\ s(Q)(I)\}$.

6. - The addition theorem for W.

In this section we prove the first theorem stated in the Introduction. We begin with a lemma.

Lemma 6.1. *Suppose* B *is a bounded element of* R *and* $0 \leqslant K$. *Suppose that for each* V *in* F,

1) $L(V)$ *and* $G(V)$ *are, respectively, the* sup *and* inf *of the set of sums* $\sum_D B(I)\, m(I)$, *where* D *is a subdivision of* V.

2) $L^*(V)$ *and* $G^*(V)$ *are, respectively, the* sup *and* inf *of the set of sums* $\sum_D \min\{K,\ B(I)\}\, m(I)$, *where* D *is a subdivision of* V.

Then

$$\int_U L^*(V) = \int_U \min\{K\, m(V),\ L(V)\}, \qquad \int_U G^*(V) = \int_U \min\{K\, m(V),\ G(V)\}.$$

Proof. Suppose $0 < c$.

There is a subdivision D of U such that if E is a refinement of D, then $|\int_U \min\{K\, m(V),\ L(V)\} - \sum_E \min\{K\, m(V),\ L(V)\}| < c/4$ and $|\int_U L^*(V) - \sum_E L^*(V)\,| < c/4$.

For each V in D, there is a subdivision S_V of V such that $0 \leqslant L^*(V) - \sum_{S_V} \min\{K\, m(I),\ B(I)\, m(I)\} < c/(4M)$, where M is the number of elements of D, so that $0 \leqslant \sum_{S_V} (L^*(I) - \min\{K\, m(I), B(I)\, m(I)\}) < c/(4M)$ and therefore $0 \leqslant \sum_D \sum_{S_V} (L^*(I) - \min\{K\, m(I),\ B(I)\, m(I)\}) < c/4$, which implies that

$$-c/2 < \int_U L^*(V) - [\textstyle\sum_D \sum_{S_V} \min\{K\, m(I),\ B(I)\, m(I)\}]_1 < c/2.$$

Now

$$[\ \ ]_1 \leqslant [\textstyle\sum_D \sum_{S_V} \min\{K\, m(I),\ L(I)\}]_2$$

and $\quad -c/4 < [\ ]_2 - \int_U \min\{K\, m(V),\ \ L(V)\} < c/4.$ Therefore $-3c/4 < [\ ]_2 - [\ ]_1 - [\int_U \min\{K\, m(V),\ \ L(V)\} - \int_U L^*(V)] < 3c/4.$

Now, for each V in D there is a subdivision S'_V of V such that $0 \leqslant L(V) - \sum_{S'_V} B(I)\, m(I) < c/(4M)$ so that

$$0 \leqslant [\sum_D \sum_{S'_V} \{L(I) - B(I)\, m(I)\}]_3 < c/4$$

and therefore

$$0 \leqslant [\sum_D \sum_{S'_V} \min\{K\, m(I),\ L(I)\}]_4 - [\sum_D \sum_{S'_V} \min\{K\, m(I),\ B(I)\, m(I)\}]_5 \leqslant [\ ]_3 < c/4.$$

$$[\ ]_5 \leqslant [\sum_D \sum_{S'_V} L^*(I)]_6,$$

$-c/4 < \int_U L^*(V) - [\ ]_6 < c/4$ and $-c/4 < \int_U \min\{K\, m(V), L(V)\} - [\ ]_4 < c/4.$ Therefore $-3c/4 < [\ ]_5 - [\ ]_4 + \int_U L^*(V) - [\ ]_6 + [\ ]_4 - \int_U \min\{Km(V), L(V)\} < 3c/4,$ so that $-3c/4 < [\ ]_5 - [\ ]_6 - [\int_U \min\{K\, m(V),\ \ L(V)\} - \int_U L^*(V)] < 3c/4.$

Therefore $\int_U L^*(V) \leqslant \int_U \min\{K\, m(V),\ L(V)\} \leqslant \int_U L^*(V).$

Therefore $\int_U \min\{K\, m(V),\ \ L(V)\} = \int_U L^*(V).$

In a similar fashion it follows that $\int_U G^*(V) = \int_U \min\{K\, m(V),\ G(V)\}.$

We now prove Theorem 6.1, as stated in the Introduction.

P r o o f. We first observe that if each of X, X', X^*, A and B is a number such that $0 \leqslant \min\{A,\ B,\ X,\ \min\{X',\ X^*\} - X\}$, then $\min\{X,\ A + B\} = \min\{X,\ \min\{X',\ A\} + \min\{X^*,\ B\}$

First, suppose P is in W.

Suppose each of K, K' and K^* is a number such that $0 \leqslant \min\{K, \min\{K',\ K^*\} - K\}$, and V is in F.

$$[\int_V \min\{K\, m(I),\ \int_I \min\{K',\ \ H(J)\}\, m(J) + \int_I \min\{K^*,\ P(J)\}\ m(J)\}]_1 =$$

$$= \int_V \min\{K\, m(I),\ \min\{K',\ H(I)\}\, m(I) + \min\{K^*,\ P(I)\}\ m(I)\} =$$

$$= \int_V \min\{K,\ \min\{K',\ H(I)\} + \min\{K^*,\ P(I)\}\}\ m(I) =$$

$$= \int_V \min\{K,\ H(I) + P(I)\}\, m(I).$$

Now $0 \leqslant \int_V \min\{K\, m(I),\ \ s(H)(I) + s(P)(I)\} - [\ ]_1 \leqslant s(H)(V) + s(P)(V) -$

$[\int_V \min\{K', H(I)\} m(I) + \int_V \min\{K^*, P(I)\} m(I)] \to 0$ as $\min\{K', K^*\} \to \to \infty$. Therefore

$$\int_V \min\{K\, m(I),\ s(H)(I) + s(P)(I)\} = \int_V \min\{K,\ H(I) + P(I)\} m(I) .$$

From Lemma 5.1 it follows that $s(H) + s(P)$ is in C, so that from Theorem 5.A, $\int_V \min\{K,\ H(I) + P(I)\} m(I) = \int_V \min\{K\, m(I),\ s(H)(I) + s(P)(I)\} \to \to s(H)(V) + s(P)(V)$ as $K \to \infty$.

Therefore $H + P$ is in W and $s(H + P) = s(H) + s(P)$.

Now suppose that $H + P$ is in W.

Suppose each of K, K' and K^* is a number and $0 \leqslant \min\{K,\ \min\{K', K^*\} - K\}$.

$$\int_U \min\{K,\ H(V) + P(V)\} m(V) =$$

$$= [\int_U \min\{K\, m(V),\ \min\{K', H(V)\} m(V) + \min\{K^*, P(I)\}\} m(V)]_2 .$$

If D is a subdivision of U, then

$$|\textstyle\sum_D \min\{K\, m(V),\ \min\{K', H(V)\} m(V) + \min\{K^*, P(V)\} m(V)\} -$$

$$[\textstyle\sum_D \min\{K\, m(V),\ [\int_V \min\{K', H(I)\} m(I)] + \min\{K^*, P(V)\} m(V)\}]_3 | \leqslant$$

$$\leqslant \textstyle\sum_D | \min\{K', H(V)\} m(V) - \int_V \min\{K', H(I)\} m(I) | ,$$

so that, by Theorem 2.K.,

$$[\int_U \min\{K\, m(V),\ [\int_V \min\{K', H(I)\} m(I)] + \min\{K^*, P(V)\} m(V)\}]_4$$

exists and is $[\ ]_2$.

Again, if D is a subdivision of U, then

$$0 \leqslant [\textstyle\sum_D \min\{K\, m(V),\ s(H)(V) + \min\{K^*, P(V)\} m(V)\}]_5 - [\ ]_3 \leqslant$$

$$\leqslant \textstyle\sum_D [s(H)(V) - \int_V \min\{K', H(I)\} m(I)] = s(H)(U) - \int_U \min\{K', H(V)\}\ m(V) .$$

Now suppose $c > 0$. There is a number $K' > K$ such that $0 \leqslant s(H)(U) - \int_U \min\{K', H(V)\} m(V) < c/2$, so that $0 < [\ ]_5 - [\ ]_3 < c/2$. There is a subdivision E of U such that if D is a refinement of E, then $c/2 > | [\ ]_3 - [\ ]_4 | = = | [\ ]_3 - \int_U \min\{K,\ H(V) + P(V)\} m(V) |$, and therefore $| [\ ]_5 - \int_U \min\{K,\ H(V) + P(V)\} m(V) | < c$.

Therefore

$$[\int_U \min\{K\, m(V),\ s(H)(V) + \min\{K^*,\ P(V)\}\, m(V)\}]_6$$

exists and is $\int_U \min\{K,\ H(V) + P(V)\}\, m(V)$.

For each V in F and $K'' \geqslant 0$, we let

$$L(K'')(V) \quad \text{and} \quad G(K'')(V)\,,$$

denote respectively the sup and inf of the set of all sums $\sum_E \min\{K'',\ P(I)\}\, m(I)$, where E is a subdivision of V.

We show that

$$[\int_U \min\{K\, m(V),\ s(H)(V) + L(K^*)(V)\}]_7 = [\ ]_6\,.$$

Suppose $c > 0$. There is a subdivision D of U such that if E' is a refinement of D, then $|[\ ]_7 - \sum_{E'} \min\{K\, m(V),\ s(H)(V) + L(K^*)(V)\}| < c/4$ and $|[\ ]_6 - \sum_{E'} \min\{K\, m(V),\ s(H)(V) + \min\{K^*,\ P(V)\}\, m(V)\}| < c/4$. For each V in D, there is a subdivision S_V of V such that $0 \leqslant L(K^*)(V) - \sum_{S_V} \min \{K^*,\ P(I)\}\, m(I) < c/(4M)$, where M is the number of elements of D, so that

$$0 \leqslant [\sum_D \sum_{S_V} \{L(K^*)(I) - \min\{K^*,\ P(I)\}\, m(I)\}]_8 < c/4.$$

Now

$$|[\ ]_7 - [\sum_D \sum_{S_V} \min\{K\, m(I),\ s(H)(I) + L(K^*)(I)\}]_9| < c/4\,.$$

$$|[\ ]_6 - [\sum_D \sum_{S_V} \{\min\{K\, m(I),\ s(H)(I) + \min\{K^*,\ P(I)\}\, m(I)\}]_{10}| < c/4\,.$$

$0 \leqslant [\ ]_9 - [\ ]_{10} \leqslant [\ ]_8 < c/4$. Therefore $|[\ ]_7 - [\ ]_6| < 3c/4$.

Therefore $[\ ]_7 = [\ ]_6$.

In a similar fashion it follows that $\int_U \min\{K\, m(V),\ s(H)(V) + G(K^*)(V)\} = [\ ]_6$.

Suppose V is in F. We see that if $0 \leqslant K''$, then $\int_V G(K'')(I) \leqslant \int_V L(K'')\,(I) \leqslant s(H+P)(V)$. Furthermore, if $0 \leqslant K'' \leqslant K''^*$, then $\int_V L(K'')(I) \leqslant \int_V L(K''^*)(I)$ and $\int_V G(K'')(I) \leqslant \int_V G(K''^*)(I)$.

For each V in F, we let

$$h(V) \quad \text{and} \quad g(V)$$

denote, respectively,

$$\sup \int_V L(K'')(I) \quad \text{and} \quad \sup \int_V G(K'')(I) ,$$

for $0 \leqslant K''$.

We see that each of h, g, $s(H+P)-h$ and $s(H+P)-g$ is in R_A^+, so that each of h and g is in C. Furthermore,

$$\int_U \left[|h(V) - \int_V L(K'')(I)| + |g(V) - \int_V G(K'')(I)| \right] \to 0 \quad \text{as} \quad K'' \to \infty .$$

Now

$$|[\int_U \min\{K\, m(V),\ s(H)(V) + h(V)\}]_{11} - \int_U \min\{K,\ H(V) + P(V)\} m(V)| =$$

$$= |[\ ]_{11} - [\ ]_7| \leqslant h(U) - \int_U L(K^*)(V) \to 0 \qquad \text{as} \quad K^* \to \infty .$$

Therefore $\int_U \min\{K\, m(V),\ s(H)(V) + h(V)\} = \int_U \min\{K,\ H(V) + P(V)\}$ $m(V)$.

In a similar manner it follows that

$$\int_U \min\{K\, m(V),\ s(H)(V) + g(V)\} = \int_U \min\{K,\ H(V) + P(V)\} m(V) .$$

We see that each of $s(H) + h$ and $s(H) + g$ is in C. It therefore follows from Theorem 5.A., letting $K \to \infty$, that $s(H)(U) + h(U) = s(H+P)(U) = s(H)(U) + g(U)$, so that $h(U) = g(U)$.

Now, if $0 \leqslant K'' \leqslant K''^*$, then by Lemma 6.1 and inequality (2.6), $\int_U L(K'')(V) - \int_U G(K'')(V) = \int_U \min\{K''\, m(V), \int^V L(K''^*)(I)\} - \int_U \min\{K''\, m(V), \int_V G(K''^*)(I)\} \leqslant \int_U L(K''^*)(V) - \int_U G(K''^*)(V) \to h(U) - g(U)$ as $K''^* \to \infty$. But $h(U) - g(U) = 0$. Therefore $\int_U L(K'')(V) = \int_U G(K'')(V)$, and therefore $\int_U \min\{K'', P(V)\} m(V)$ exists. Furthermore, for each $K'' \geqslant 0$, $\int_U \min\{K'', P(V)\} m(V) \leqslant \int_U \min\{K'', H(V) + P(V)\} m(V) \leqslant s(H+P)(U)$, so that P is in W.

Therefore $H + P$ is in W if and only if P is in W, in which case

$$s(H+P) = s(H) + s(P) .$$

7. - Two product theorems for W.

Suppose P is in W.

Lemma 7.1. *If* $0 \leqslant c$, *then* cP *is in* W *and* $s(cP) = c\,s(P)$.

Proof. The lemma is obvious for $c = 0$.

Suppose $0 < c$. If V is in F and $0 \leqslant K$, then $c \int_V \min\{K/c, H(I)\}\cdot m(I) = \int_V \min\{K,\ c\,H(I)\}\,m(I)$. Letting $K \to \infty$, we see that the lemma follows.

We state the following lemma.

Lemma 7.2. *If H is in R^+ and is bounded, then H is in W if and only if* $\int_U H(V)\,m(V)$ *exists, in which case, for each V in F,* $\int_V H(I)\,m(I) = s(H)(V)$.

Lemma 7.3. *If H is in W and is bounded and* $0 < K$, *then*

$$\int_U \min\{K\,H(V),\ P(V)\}\,m(V)$$

exists.

Proof. There is a number $K^* > 0$ such that $1 - H/K^*$ is in R^+. Let $K' = KK^*$. $K'H/K^*$ is in W and bounded. Therefore, by Lemma 7.2, $\int_U \min\{K'\,H(V)/K^*,\ \min\{K',\ P(V)\}\}\,m(V)$ exists and is $\int_U \min\{K'\,H(V)/K^*,\ K',\ P(V)\}\,m(V) = \int_U \min\{K'\,H(V)/K^*,\ P(V)\}\,m(V) = \int_U \min\{K\,H(V), P(V)\}\cdot m(V)$.

Theorem 7.1. *If H is in W and bounded, then* HP *is in* W, *and for each V in F,*

$$s(H\,P)(V) = \int_V H(I)\,s(P)(I).$$

Proof. There is a number M such that $M - H$ is in R^+.

We first show that HP is in W. Suppose $0 < K$ and V is in F.

Suppose $0 < d$. $1/(H + d)$ is bounded, and by Theorem 4.3 $\int_V m(I)/(H(I) + d)$ exists. By Lemma 7.3, $\int_V \min\{K/(H(I) + d),\ P(I)\}\,m(I)$ exists, so that by Theorem 4.1, $\int_V (H(I) + d) \min\{K/(H(I) + d),\ P(I)\}\,m(I)$ exists and is $\int_V \min\{K,\ (H(I) + d)\,P(I)\}\,m(I)$.

Now let T_K denote

$$\inf \int_V \min\{K,\ (H(I) + d)\,P(I)\}\,m(I)\,,$$

for $0 < d$.

Suppose $0 < c$. There is a number $d^* > 0$ such that $c/3 > d^* s(P)(V) = s(d^* P)(V) \geqslant \int_V \min\{K,\ d^* P(I)\} m(I)$.

There is a number $d' > 0$ such that $\int_V \min\{K,\ (H(I) + d')\, P(I)\} m(I) - T_K < c/6$.

Let $d'' = \min\{d',\ d^*\}$.

We see that there is a subdivision D of V such that if E is a refinement of D, then $\sum_E \min\{K,\ d''\, P(I)\} m(I) < c/3$ and

$$| T_K - [\textstyle\sum_E \min\{K,\ (H(I) + d'')\, P(I)\} m(I)]_1 | < c/3 ,$$

so that

$$| T_K - [\textstyle\sum_E \min\{K,\ H(I)\, P(I)\} m(I)]_2 | \leqslant | T_K - [\ ]_1 | + | [\ ]_1 - [\ ]_2 | <$$

$$< c/3 + \textstyle\sum_E \min\{K,\ d'' P(I)\} m(I) < c/3 + c/3 ,$$

by inequality (2.5).

Therefore $\int_V \min\{K,\ H(I)\, P(I)\} m(I)$ exists and is T_K. Furthermore, $\int_V \min\{K,\ H(I)\, P(I)\} m(I) \leqslant \int_V \min\{K,\ M\, P(I)\} m(I) \leqslant s(M\, P)(V)$, so that $H\, P$ is in W.

We now prove the remainder of the theorem.

Again, suppose V is in F.

Suppose $0 \leqslant d$. $H + d$ is in W.

If K'' is a positive number, then, by Theorem 4.1 and Corollary 2.K., $\int_V (H(I) + d) \int_I \min\{K'',\ P(J)\} m(J)$ exists. By Theorem 4.4 and Lemma 5.1. $\int_V (H(I) + d)\, s(P)(I)$ exists and

$$\int_V (H(I) + d) \int_I \min\{K'',\ P(J)\} m(J) \to \int_V (H(I) + d)\, s(P)(I) \quad \text{as} \quad K'' \to \infty .$$

Suppose $0 < K$ and $0 < c$.

$$\int_V (H(I) + c) \min\{K/(M + c),\ P(I)\} m(I) \leqslant$$

$$\leqslant \int_V \min\{K,\ (H(I) + c)\, P(I)\} m(I) \leqslant \int_V (H(I) + c) \min\{K/c,\ P(I)\} m(I).$$

Therefore $\int_V (H(I) + c)\, s(P)(I) \leqslant s((H + c)\, P)(V) \leqslant \int_V (H(I) + c)\ s(P)(I)$, so that $s((H + c)P)(V) = \int_V (H(I) + c)\, s(P)(I)$.

Now $s(H\, P)(V) + s(c\, P)(V) = s((H + c)P)(V) = \int_V (H(I) + c)\ s(P)(I) = \int_V H(I)\, s(P)(I) + \int_V c\, s(P)(I) = \int_V H(I)\ s(P)(I) + s(c\, P)(V)$.

Therefore $s(H\, P)(V) = \int_V H(I)\, s(P)(I)$.

8. - $\min\{H, P\}$ and $\max\{H, P\}$ for H and P in W.

Theorem 8.1. *If each of H and P is in W, then each of* $\max\{H, P\}$ *and* $\min\{H, P\}$ *is in W and, for each V in F,*

$$s(\max\{H, P\})(V) = \int_V \max\{s(H)(I), s(P)(I)\}$$

and

$$s(\min\{H, P\})(V) = \int_V \min\{s(H)(I), s(P)(I)\}.$$

Proof. Suppose V is in F.
If $0 \leqslant K$, then

$$\int_V \max\left\{\int_I \min\{K, H(J)\}m(J), \int_I \min\{K, P(J)\}m(J)\right\} =$$

$$= \int_V \max\left\{\min\{K, H(I)\}m(I), \min\{K, P(I)\}m(I)\right\} =$$

$$= \int_V \min\left\{K, \max\{H(I), P(I)\}\right\}m(I).$$

Furthermore, $0 \leqslant \int_V \max\{s(H)(I), s(P)(I)\} - \int_V \max\left\{\int_I \min\{K, H(J)\}\cdot m(J), \int_I \min\{K, P(J)\}m(J)\right\} \leqslant \int_V |s(H)(I) - \int_I \min\{K, H(J)\}m(J)| + |s(P)(I) - \int_I \min\{K, P(J)\}m(J)| \to 0$ as $K \to \infty$ by Lemma 5.1. Therefore $\max\{H, P\}$ is in W and

$$s(\max\{H, P\})(V) = \int_V \max\{s(H)(I), s(P)(I)\}.$$

If $0 \leqslant K$, then

$$\int_V \min\left\{\int_I \min\{K, H(J)\}m(J), \int_I \min\{K, P(J)\}m(J)\right\} =$$

$$= \int_V \min\left\{\min\{K, H(I)\}m(I), \min\{K, P(I)\}m(I)\right\} =$$

$$= \int_V \min\left\{K, \min\{H(I), P(I)\}\right\}m(I),$$

so that by a procedure similar to that in the preceding paragraph, $\min\{H, P\}$ is in W and

$$s(\min\{H, P\})(V) = \int_V \min\{s(H)(I), s(P)(I)\}.$$

9. - $H^q P^{1-q}$ for H and P in W and $0<q<1$.

In this section we prove Theorem 9.1, the second analogue of a theorem of HENSTOCK [5] mentioned in the Introduction.

We begin with a lemma.

Lemma 9.1. *If Q is in W and $0<t<1$, then Q^t is in W, and for each V in F,*

$$s(Q^t)(V) = \int_V s(Q)(I)^t m(I)^{1-t}.$$

Proof. Suppose $0 \leqslant K$ and V is in F.

$$\int_V [\min\{K^{1/t}, Q(I)\} m(I)]^t m(I)^{1-t} = \int_V [\min\{K^{1/t}, Q(I)\}]^t m(I) =$$

$$= \int_V \min\{K, Q(I)^t\} m(I).$$

Now $\int_U | \int_V s(Q)(I)^t m(I)^{1-t} - \int_V \min\{K, Q(I)^t\} m(I) | \leqslant \int_U \int_V [s(Q)(I) - \int_I \min\{K^{1/t}, Q(J)\} m(J)]^t m(I)^{1-t} \leqslant [s(Q)(U) - \int_U \min\{K^{1/t}, Q(V)\} m(V)]^t \cdot m(U)^{1-t} \to 0$ as $K \to \infty$, so that Q^t is in W and for each V in F,

$$s(Q^t)(V) = \int_V s(Q)(I)^t m(I)^{1-t}.$$

We now prove Theorem 9.1, as stated in the Introduction.

Proof. Suppose $0<K$ and V is in F.

Suppose $0<c$ and each of K' and K^* is a number such that $\min\{K', K^*\} \geqslant \max\{K, (K/c^{1-q})^{1/q}, (K/c^q)^{1/(1-q)}\}$.

$$\int_V \min\{K\, m(I), [\min\{K', H(I)+c\} m(I)]^q [\min\{K^*, P(I)+c\} m(I)]^{1-q}\} =$$

$$= \int_V \min\{K, \min\{K'^q, (H(I)+c)^q\} \min\{K^{*1-q}, (P(I)+c)^{1-q}\}\} m(I) =$$

$$= \int_V \min\{K, K'^q K^{*1-q}, K'^q (P(I)+c)^{1-q}, (H(I)+c)^q K^{*1-q},$$

$$(H(I)+c)^q (P(I)+c)^{1-q}\} m(I) =$$

$$= \int_V \min\{K, (H(I)+c)^q (P(I)+c)^{1-q}\} m(I),$$

since, for each I in F, $\min\{K'^q K^{*1-q}, K'^q(P(I)+c)^{1-q}, (H(I)+c)^q K^{*1-q}\} \geqslant K$.

Let T_K denote

$$\inf \int_V \min\{K,\ (H(I)+c)^q\,(P(I)+c)^{1-q}\}\,m(I)\,,$$

for $0 < c$.

Suppose $0 < d$.

There is a number $c' > 0$ such that

$$0 \leqslant \int_V \min\{K,\ (H(I)+c')^q\,(P(I)+c')^{1-q}\}\,m(I) - T_k < d/4.$$

There is a number $c^* > 0$ such that

$$c^{*1-q}\,s([H+c^*]^q)(U) + c^{*q}\,s(P^{1-q})(U) < d/4.$$

Let $c'^* = \min\{c',\ c^*\}$.

By Lemmas 7.1 and 9.1 and Theorem 6.1,

$$[\int_V \min\{K,\ (H(I)+c'^*)^q\,c'^{*1-q} + c'^{*q}\,P(I)^{1-q}\}\,m(I)]_1$$

exists, so that there is a subdivision E' of V such that if E is a refinement of E', then

$$|\,[\ ]_1 - [\textstyle\sum_E \min\{K,\ (H(I)+c'^*)^q\,c'^{*1-q} + c'^{*q}\,P(I)^{1-q}\}\,m(I)]_2\,| < d/4.$$

There is a subdivision E^* of V such that if E is a refinement of E^*, then

$$|\,[\int_V \min\{K,\ (H(I)+c'^*)^q\,(P(I)+c'^*)^{1-q}\}\,m(I)]_3 -$$

$$[\textstyle\sum_E \min\{K,\ (H(I)+c'^*)^q\,(P(I)+c'^*)^{1-q}\}\,m(I)]_4\,| < d/4\,.$$

There is a common refinement E'^* of E' and E^*.

Suppose E is a refinement of E'^*.

$$0 \leqslant [\ ]_4 - [\textstyle\sum_E \min\{K,\ H(I)^q\,P(I)^{1-q}\}\,m(I)]_5 \leqslant$$

$$\leqslant \textstyle\sum_E \min\{K,\ (H(I)+c'^*)^q\,(P(I)+c'^*)^{1-q} - H(I)^q\,P(I)^{1-q}\}\,m(I) \leqslant$$

$$\leqslant [\ ]_2 < d/4 + [\ ]_1 \leqslant d/4 + c'^{*1-q}\,s([H+c'^*]^q)(U) +$$

$$+ c'^{*q}\,s(P^{1-q})(U) < d/4 + d/4 = d/2.$$

Since $|[\ ]_3 - [\ ]_4| < d/4$, it follows that $|[\ ]_3 - [\ ]_5| < 3d/4$. Since, $0 \leqslant [\ ]_3 - T_K < d/4$, it follows that $|T_K - [\ ]_5| < d$. Therefore $\int_V \min\{K, H(I)^q P(I)^{1-q}\} m(I)$ exists and is T_K.

Again, suppose $0 < c$ and each of K' and K^* is a number such that $\min\{K', K^*\} \geqslant \max\{K, (K/c^{1-q})^{1/q}, (K/c^q)^{1/(1-q)}\}$.

$$[\int_V \min\{K\, m(I),\ s(H + c)(I)^q\, s(P + c)(I)^{1-q}\}]_6 -$$

$$\int_V \min\{K,\ (H(I) + c)^q\, (P(I) + c)^{1-q}\} m(I) = [\ ]_6 -$$

$$\int_V \min\{Km(I), [\int_I \min\{K', H(J) + c\} m(J)]^q [\int_I \min\{K^*, P(J) + c\} m(J)]^{1-q}\} \leqslant$$

$$\leqslant \int_V | s(H + c)(I)^q\, s(P + c)(I)^{1-q} -$$

$$[\int_I \min\{K', H(J) + c\} m(J)]^q [\int_I \min\{K^*, P(J) + c\} m(J)]^{1-q} | \to 0$$

as $\min\{K', K^*\} \to \infty$ by Theorem 2.3, so that

$$\int_V \min\{K\, m(I),\ s(H + c)(I)^q\, s(P + c)(I)^{1-q}\} =$$

$$= \int_V \min\{K(H(I) + c)^q\, (P(I) + c)^{1-q}\} m(I) \to$$

$$\to \int_V \min\{K,\ H(I)^q\, P(I)^{1-q}\} m(I)$$

as $c \to 0$ by the preceding paragraphs. Furthermore,

$$0 \leqslant \int_V \min\{K\, m(I),\ s(H + c)(I)^q\, s(P + c)(I)^{1-q}\} -$$

$$\int_V \min\{K\, m(I),\ s(H)(I)^q\, s(P)(I)^{1-q}\} \leqslant$$

$$\leqslant \int_V | s(H + c)(I)^q\, s(P + c)(I)^{1-q} - s(H)(I)^q\, s(P)(I)^{1-q} | \leqslant$$

$$\leqslant \int_V s(H + c)(I)^q\, [c\, m(I)]^{1-q} + [c\, m(I)]^q\, s(P)(I)^{1-q} \leqslant$$

$$\leqslant s(H + c)(V)^q\, [c\, m(V)]^{1-q} + [c\, m(V)]^q\, s(P)(V)^{1-q} \to 0$$

as $c \to 0$.

Therefore $\int_V \min\{K\, m(I), s(H)(I)^q\, s(P)(I)^{1-q}\} = \int_V \min\{K, H(I)^q\, P(I)^{1-q}\} m(I)$.

We see that the function h on F defined by $h(V) = \int_V s(H)(I)^q\, s(P)(I)^{1-q}$ is in C. Therefore, by Theorem 5.A., $\int_V \min\{K,\ H(I)^q\, P(I)^{1-q}\} m(I) = = \int_V \min\{K\, m(I),\ \int_I s(H)(J)^q\, s(P)(J)^{1-q}\} \to \int_V s(H)(I)^q\, s(P)(I)^{1-q}$ as $K \to \infty$.

Therefore $H^q P^{1-q}$ is in W and, for each V in F, $s(H^q P^{1-q})(V) = \int_V s(H)(I)^q s(P)(I)^{1-q}$.

10. - H^q for H in W and $1 < q$.

We now prove Theorem 10.1, as stated in the Introduction.

Proof. Suppose $0 < K$.

Since $\int_U \min\{K^{1/q}, H(V)\} m(V)$ exists, it follows from Theorem 4.2 that if V is in F, then $\int_V (\min\{K^{1/q}, H(I)\})^q m(I)$ exists and is $\int_V \min\{K, H(I)^q\} \cdot m(I)$.

Now, if V is in F and D is a subdivision of V, then

$$\begin{aligned}
&| \textstyle\sum_D [\min\{K^{1/q}, H(I)\}]^q m(I) - \sum_D [\int_I \min\{K^{1/q}, H(J)\} m(J)]^q m(I)^{1-q} | = \\
&= | \textstyle\sum_D ([\min\{K^{1/q}, H(I)\} m(I)]^q - [\int_I \min\{K^{1/q}, H(J)\} m(J)]^q) m(I)^{1-q} | \leqslant \\
&\leqslant \textstyle\sum_D q [K^{1/q} m(I)]^{q-1} | \min\{K^{1/q}, H(I)\} m(I) - \int_I \min\{K^{1/q}, H(J)\} m(J) | m(I)^{1-q} \leqslant \\
&\leqslant q K^{1/q} \textstyle\sum_D | \min\{K^{1/q}, H(I)\} m(I) - \int_I \min\{K^{1/q}, H(J)\} m(J) | ,
\end{aligned}$$

so that by Theorem K, $\int_V [\min\{K^{1/q}, H(I)\}]^q m(I) = \int_V [\int_I \min\{K^{1/q}, H(J)\} \cdot m(J)]^q m(I)^{1-q}$.

If $\int_U s(H)(V)^q m(V)^{1-q}$ exists, then for each V in F, $\int_V \min\{K, H(I)^q\} \cdot m(I) = \int_V [\int_I \min\{K^{1/q}, H(J)\} m(J)]^q m(I)^{1-q} \to \int_V s(H)(I)^q m(I)^{1-q}$ as $K \to \infty$, by Theorem 3.1, so that H^q is in W and for each V in F, $s(H^q)(V) = \int_V s(H)(I)^q \cdot m(I)^{1-q}$.

Now suppose H^q is in W. For each positive integer n and each V in F,

$$\int_V [\int_I \min\{n^{1/q}, H(J)\} m(J)]^q m(I)^{1-q} = \int_V \min\{n, H(I)^q\} m(I) \to s(H^q)(V)$$

as $n \to \infty$, so that again by Theorem 3.1, $\int_V s(H)(I)^q m(I)^{1-q}$ exists and is $s(H^q)(V)$.

11. - $1/H$ for H in W.

We now prove a special case of the last theorem mentioned in the Introduction.

Theorem 11.1. *If H is in W and for some number $T > 0$, $H - T$ is in R^+, then, for each V in F, $\int_V m(I)/H(I)$ exists and is $\int_V m(I)^2/s(H)(I)$.*

Proof. Suppose V is in F.

If $0 < K$, then, by Theorem 4.3, $\int_V m(I)/\min\{K, H(I)\}$ exists and is $\int_V m(I)^2/\int_I \min\{K, H(J)\} m(J)$.

Let

$$S = \inf \int_V m(I)/\min\{K, H(I)\},$$

for $0 < K$.

Suppose $0 < c$. There is a number $K' > 0$ such that $m(V)/K' < c/4$ and $0 \leqslant \int_V m(I)/\min\{K', H(I)\} - S < c/4$. There is a subdivision D of V such that if E is a refinement of D, then

$$\left| \int_V m(I)/\min\{K', H(I)\} - \sum_E m(I)/\min\{K', H(I)\} \right| < c/4,$$

so that $| S - \sum_E m(I)/\min\{K', H(I)\}| < c/2$; furthermore $| \sum_E m(I)/\min\{K', H(I)\} - \sum_E m(I)/H(I) | = | \sum_E \max\{1/K', 1/H(I)\} m(I) - m(I)/H(I) | \leqslant \leqslant \sum_E m(I)/K' = m(V)/K' < c/4$, so that $| S - \sum_E m(I)/H(I) | < 3c/4$. Therefore $S = \int_V m(I)/H(I)$.

We therefore see that if $0 < K$, then $\int_V m(I)^2/\int_I \min\{K, H(J)\} m(J) = = \int_V m(I)/\min\{K, H(I)\} \to \int_V m(I)/H(I)$ as $K \to \infty$. By Theorem 3.2,

$$\int_V m(I)^2/\int_I \min\{K, H(J)\} m(J) \to \int_V m(I)^2/s(H)(I) \quad \text{as} \quad K \to \infty.$$

Therefore $\int_V m(I)/H(I) = \int_V m(I)^2/s(H)(I)$.

12. - The set W^*.

We begin by stating an immediate consequence of Theorem 6.1, as mentioned in the Introduction.

Lemma 12.1. *If each of H', H'', P' and P'' is in W and $H' - P' = = H'' - P''$, then $s(H') - s(P') = s(H'') - s(P'')$.*

We now prove Theorem 12.1, as stated in the Introduction.

Proof. Obviously $W \subseteq W^* \cap R^+$.

Suppose H is in $W^* \cap R^+$. Then for some P and P', each in W, $H = = P - P'$, so that $P = H + P'$ which, by Theorem 6.1 implies that H is in W.

The final statement of Theorem 12.1 is again an immediate consequence of Theorem 6.1.

As stated in the Introduction, the consistency implied by Lemma 12.1 and Theorem 12.1 permits the following extension of the function s from W to W^*:

If H is in W^*, then we define $S(H)$ to be the function x on F such that if each of Y and Y' is in W and $H = Y - Y'$, then $x = s(Y) - s(Y')$.

We see that if H is in W^*, then $s(H)$ is in C and is bounded.

We state the next theorem without proof:

Theorem 12.2. *If each of a and b is a number and each of H and P is in W^*, then $aH + bP$ is in W^* and $s(aH + bP) = a\,s(H) + b\,s(P)$.*

Theorem 12.3. *If H is in W^*, then $|H|$ is in W and, for each V in F,*

$$s(|H|)(V) = \int_V |s(H)(I)|.$$

Proof. For some P and P', each in W, $H = P - P'$, so that $|H| = |P - P'| = \max\{P, P'\} - \min\{P, P'\}$, which is in W by Theorems 8.1 and 12.1, which imply further that for each V in F, $s(|H|)(V) = s(\max\{P, P'\})(V) - s(\min\{P, P'\})(V) = \int_V \max\{s(P)(I), s(P')(I)\} - \int_V \min\{s(P)(I), s(P')(I)\} = \int_V |s(P)(I) - s(P')(I)| = \int_V |s(P - P')(I)| = \int_V |s(H)(I)|$.

In a similar fashion, by means of successive applications of the formulas: $\min\{a, b\} = (a + b - |a - b|)/2$ and $\max\{a, b\} = |a - b| + \min\{a, b\}$, we have the following analogue of a theorem of Henstock [5] mentioned in the Introduction which we state without proof.

Theorem 12.4. *If each of H and P is in W^*, then each of $\min\{H, P\}$ and $\max\{H, P\}$ is in W^*, and, for each V in F,*

$$s(\min\{H, P\})(V) = \int_V \min\{s(H)(I), s(P)(I)\}$$

and

$$s(\max\{H, P\})(V) = \int_V \max\{s(H)(I), s(P)(I)\}.$$

Lemma 12.2. *If H is in R and bounded, then H is in W^* if and only if $\int_U H(V)\,m(V)$ exists, in which case, for each V in F,*

$$s(H)(V) = \int_V H(I)\,m(I).$$

Proof. First suppose that H is in W^*. If V is in F, then $s(H)(V) = s(\max\{H, 0\} + \min\{H, 0\})(V) = s(\max\{H, 0\})(V) - s(-\min\{H, 0\})(V) = \int_V \max\{H(I), 0\}m(I) - \int_V -\min\{H(I), 0\}m(I) = \int_V H(I)\,m(I)$, by Lemma 7.2.

Now suppose that $\int_U H(V)\,m(V)$ exists. We see that each of $\int_U \max\{H(V), 0\}m(V)$ and $\int_U \min\{H(V), 0\}m(V)$ exists, so that each of $\max\{H, 0\}$ and $-\min\{H, 0\}$ is in W, so that H is in W^*.

Theorem 12.5. *If each of H and P is in W^* and H is bounded, then HP is in W^* and, for each V in F,*

$$s(HP)(V) = \int_V H(I)\, s(P)(I)\,.$$

Proof. For some Y and Y', each in W, $P = Y - Y'$.

We see that $HP = (\max\{H,0\} + \min\{H,0\})(Y - Y') = [\max\{H,0\}Y + + -\min\{H,0\}Y'] - [\max\{H,0\}Y' + -\min\{H,0\}Y]$, so that, by Lemma 7.2 and Theorems 6.1, 7.1, 12.1 and 12.4, HP is in W^*; furthermore, if V is in F, then: $s(HP)(V) = s(\max\{H, 0\}Y)(V) + s(-\min\{H, 0\}Y')(V) - s(\max\{H, 0\}Y')(V) - s(-\min\{H, 0\}Y)(V) = \int_V \max\{H(I), 0\}s(Y)(I) + + \int_V -\min\{H(I), 0\}s(Y')(I) - \int_V \max\{H(I), 0\}s(Y')(I) - \int_V -\min\{H(I), 0\}. s(Y)(I) = \int_V H(I)[s(Y)(I) - s(Y')(I)] = \int_V H(I)\, s(P)(I)$.

We now prove Theorem 12.6, as stated in the Introduction.

Proof. By Theorem 12.3, $|H|$ is in W, so that by Theorem 11.1, $1/|H|$, which is bounded, is in W, so that $\int_U m(V)/|H(V)|$ exists. By Theorem 12.5, $H/|H|$, which is bounded, is in W^*, so that $\int_U H(V)\, m(V)/|H(V)|$ exists. By Theorem 4.1, $\int_U H(V)\, m(V)/(|H(V)|^2)$ exists and is $\int_U m(V)/H(V)$.

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Summary.

See Introduction.

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