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On a Polynomial of the Form F_1. (**)

1. - We consider a polynomial $S_n(\alpha, \beta, \gamma, x, y)$, for n a non-negative integer, defined by

(1) $$S_n(\alpha, \beta, \gamma, x, y) = \frac{(\alpha)_n}{n!} F_1(-n; \beta, \gamma; \alpha; x, y),$$

where F_1 is one of the APPELL's functions, in general, defined as

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{m!\, n!\, (\gamma)_{m+n}} x^m y^n .$$

Putting $y = x$ and using (cf. [1], p. 239)

$$F_1(\alpha; \beta, \beta'; \gamma; x, x) = {}_2F_1(\alpha, \beta+\beta'; \gamma; x),$$

(1) becomes

(2) $$S_n(\alpha, \beta, \gamma, x, x) = \frac{(\alpha)_n}{n!} {}_2F_1(-n, \beta+\gamma; \alpha; x).$$

From the definition (1) we find below the generating function.

We have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\alpha)_n} S_n(\alpha, \beta, \gamma, x, y) t^n =$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \sum_{p+q \le n} \frac{(-n)_{p+q} (\beta)_p (\gamma)_q}{p!\ q!\ (\alpha)_{p+q}} x^p y^q t^n = \sum_{n,p,q=0}^{\infty} \frac{(\lambda)_{n+p+q} (\beta)_p (\gamma)_q}{n!\, p!\, q!\, (\alpha)_{p+q}} (-xt)^p (-yt)^q t^n.$$

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Thus we arrive at the generating function

$$(3) \qquad \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\alpha)_n} S_n(\alpha, \beta, \gamma, x, y)\, t^n = (1-t)^{-\lambda} F_1\left(\lambda;\ \beta,\ \gamma;\ \alpha;\ \frac{-x\,t}{1-t},\ \frac{-y\,t}{1-t}\right).$$

Dividing t by λ and taking $\lambda \to \infty$, it follows from (3)

$$(4) \qquad \sum_{n=0}^{\infty} \frac{t^n}{(\alpha)_n} S_n(\alpha,\ \beta,\ \gamma,\ x,\ y) = e^t \varphi_2(\beta,\ \gamma;\ \alpha;\ -x\,t,\ -y\,t),$$

where

$$\varphi_2(\beta,\ \beta';\ \gamma;\ x,\ y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_m\,(\beta')_n}{m!\,n!\,(\gamma)_{m+n}} x^m y^n\,,$$

while for $\lambda = \alpha$ (3) becomes

$$(5) \qquad \sum_{n=0}^{\infty} S_n(\alpha,\ \beta,\ \gamma,\ x,\ y)\, t^n = (1-t)^{\beta+\gamma-\alpha}\,(1-t+x\,t)^{-\beta}\,(1-t+y\,t)^{-\gamma}\,.$$

2. - Special properties.

We rewrite (4) as

$$\varphi_2(\beta, \gamma;\alpha;\ -x\,t,\ -y\,t) = e^t \sum_{k=0}^{\infty} \frac{t^k}{(\alpha)_k} S_k(\alpha,\ \beta,\ \gamma,\ x,\ y)$$

or

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(\beta)_{n-r}\,(\gamma)_r}{r!\,(n-r)!\,(\alpha)_n} (-x\,t)^{n-r}\,(-y\,t)^r = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!\,(\alpha)_k} S_k(\alpha,\ \beta,\ \gamma,\ x,\ y)\, t^n\,,$$

from which we get

$$(6) \qquad {}_2F_1(-n,\ \gamma;\ 1-\beta-n;\ y/x) = \frac{n!\,(\alpha)_n}{(\beta)_n} x^{-n} \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,(\alpha)_k} S_k(\alpha,\ \beta,\ \gamma,\ x,\ y)\,.$$

Using (6) we prove:

$$(7) \qquad F_1(\lambda;\beta,\gamma;\mu;ux,uy) = \sum_{k=0}^{\infty} \frac{(-1)^k\,(\lambda)_k}{(\mu)_k} u^k {}_2F_1(\lambda+k, \alpha+k; \mu+k; u) S_k(\alpha,\beta,\gamma,x,y).$$

We have:

$$F_1(\lambda;\ \beta,\ \gamma;\ \mu;\ u\,x,\ u\,y) =$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\lambda)_{n+r}\,(\beta)_n\,(\gamma)_r}{n!\,r!\,(\mu)_{n+r}}\,(u\,x)^n\,(u\,y)^r =$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n}\,u^n\,\frac{(\beta)_n}{n!}\,x^n\ {}_2F_1(-n,\ \gamma;\ 1-\beta-n;\ y/x) =$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(\lambda)_n}{(\mu)_n}\,u^n \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,(\alpha)_k}\,S_k(\alpha,\ \beta,\ \gamma,\ x,\ y) =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k\,(\alpha)_{n+k}\,(\lambda)_{n+k}}{n!\,(\alpha)_k\,(\mu)_{n+k}}\,u^{n+k}\,S_k(\alpha,\ \beta,\ \gamma,\ x,\ y) =$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\,(\lambda)_k}{(\mu)_k}\,u^k\ {}_2F_1(\lambda+k,\ \alpha+k;\ \mu+k;\ u)\ S_k(\alpha,\ \beta,\ \gamma,\ x,\ y),$$

which completes the proof of (7).

For $y = 0$, (7) reduces to

$$(8)\quad \begin{cases} {}_2F_1(\lambda,\ \beta;\ \mu;\ u\,x) = \\ = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k\,(\lambda)_k\,(\alpha)_k}{k!\,(\mu)_k}\,u^k\ {}_2F_1(\lambda+k,\ \alpha+k;\ \mu+k;\ u)\ {}_2F_1(-n,\ \beta;\ \alpha;\ x)\,. \end{cases}$$

Next, let us consider an elementary identity

$$(1-t)^{\beta+\beta'+\gamma+\gamma'-\alpha-\alpha'}\,(1-t+x\,t)^{-\beta-\beta'}\,(1-t+y\,t)^{-\gamma-\gamma'} =$$

$$=(1-t)^{\beta+\gamma-\alpha}\,(1-t+xt)^{-\beta}\,(1-t+yt)^{-\gamma}(1-t)^{\beta'+\gamma'-\alpha'}(1-t+xt)^{-\beta'}(1-t+yt)^{-\gamma'}.$$

With the help of (5) it may be expressed as

$$\sum_{n=0}^{\infty} S_n(\alpha+\alpha',\,\beta+\beta',\,\gamma+\gamma',\,x,\,y)\,t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} S_k(\alpha,\,\beta,\,\gamma,\,x,\,y)\,S_{n-k}(\alpha',\,\beta',\,\gamma',\,x,\,y)\,t^n,$$

from which we get

$$(9)\quad S_n(\alpha+\alpha',\ \beta+\beta',\ \gamma+\gamma',\ x,\ y) = \sum_{k=0}^{n} S_k(\alpha,\beta,\gamma,x,y)\,S_{n-k}(\alpha',\beta',\gamma',x,\ y).$$

3. - More generating functions.

Consider the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k}{k} S_{n+k}(\alpha,\ \beta,\ \gamma,\ x,\ y)\, t^n v^k =$$

$$= \sum_{n=0}^{\infty} S_n(\alpha,\ \beta,\ \gamma,\ x,\ y)\, (t+v)^n =$$

$$= (1-t)^{\beta+\gamma-\alpha}\,[1+(x-1)(t+v)]^{-\beta}\,[1+(y-1)(t+v)]^{-\gamma} =$$

$$= (1-t)^{\beta+\gamma-\alpha}\,(1-t+xt)^{-\beta}\,(1-t+yt)^{-\gamma}\left(1-\frac{v}{1-t}\right)^{\beta+\gamma-\alpha}\cdot$$

$$\cdot\left[1+\left(\frac{x}{1-t+xt}-1\right)\frac{v}{1-t}\right]^{-\beta}\left[1+\left(\frac{y}{1-t+yt}-1\right)\frac{v}{1-t}\right]^{-\gamma} =$$

$$= (1-t)^{\beta+\gamma-\alpha}\,(1-t+xt)^{-\beta}\,(1-t+yt)^{-\gamma}\sum_{k=0}^{\infty} S_k\left(\alpha,\beta,\gamma,\frac{x}{1-t+xt},\frac{y}{1-t+yt}\right)\frac{v^k}{(1-t)^k},$$

from which it follows that

$$(10)\quad \begin{cases} \displaystyle\sum_{n=0}^{\infty}\binom{n+k}{k} S_{n+k}(\alpha,\ \beta,\ \gamma,\ x,\ y)\, t^n = \\[2ex] \displaystyle = (1-t)^{\beta+\gamma-\alpha-k}\,(1-t+xt)^{-\beta}\,(1-t+yt)^{-\gamma}\, S_k\left(\alpha,\beta,\gamma,\frac{x}{1-t+xt},\frac{y}{1-t+yt}\right). \end{cases}$$

By means of (10), we prove the bilinear generating function

$$(11)\quad \begin{cases} \displaystyle\sum_{n=0}^{\infty} {}_2F_1(-n,\ \lambda;\ \alpha;\ z)\, S_n(\alpha,\ \beta,\ \gamma,\ x,\ y)\, t^n = \\[2ex] = (1-t)^{\lambda+\beta+\gamma-\alpha}\,(1-t+xt)^{-\beta}\,(1-t+yt)^{-\gamma}\,(1-t+zt)^{-\lambda}\cdot \\[2ex] \displaystyle\cdot F_1\left(\lambda;\ \beta,\ \gamma;\ \alpha;\ \frac{x\,z\,t}{(1-t+xt)(1-t+zt)},\ \frac{y\,z\,t}{(1-t+yt)(1-t+zt)}\right). \end{cases}$$

Starting with the L. H. S. of (11) we have

$$\sum_{n=0}^{\infty} {}_2F_1(-n,\ \lambda;\ \alpha;\ z)\, S_n(\alpha,\ \beta,\ \gamma,\ x,\ y)\, t^n =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{\lambda_k}{(\alpha)_k} (-z)^k S_n(\alpha,\ \beta,\ \gamma,\ x,\ y)\, t^n =$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda)_k}{(\alpha)_k} (-zt)^k \sum_{n=0}^{\infty} \binom{n+k}{k} S_{n+k}(\alpha,\ \beta,\ \gamma,\ x,\ y)\, t^n =$$

$$= (1-t)^{\beta+\gamma-\alpha} (1-t+xt)^{-\beta} (1-t+yt)^{-\gamma} \cdot$$

$$\cdot \sum_{k=0}^{\infty} \frac{(\lambda)_k}{(\alpha)_k} S_k\left(\alpha,\ \beta,\ \gamma,\ \frac{x}{1-t+xt},\ \frac{y}{1-t+yt}\right) \left(\frac{-zt}{1-t}\right)^k,$$

which on using (3) yields (11).

For $y = 0$, (11) reduces to

$$(12)\quad \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!}\, {}_2F_1(-n,\ \beta;\ \alpha,\ x)\, {}_2F_1(-n,\ \lambda;\ \alpha;\ z)\, t^n = \\ \displaystyle = (1-t)^{\lambda+\beta-\alpha} (1-t+xt)^{-\beta} (1-t+zt)^{-\lambda}\, {}_2F_1\left(\lambda, \beta; \alpha; \frac{x\,z\,t}{(1-t+xt)(1-t+zt)}\right). \end{cases}$$

(12) is due to WEISNER [2].

4. - Integral representation.

Now we shall find integral representation for the product

$$S_m(\alpha,\ \beta,\ \gamma,\ x,\ x)\ S_n(\alpha',\ \beta',\ \gamma',\ y,\ y)\,.$$

By (2) we have

$$S_m(\alpha,\ \beta,\ \gamma,\ x,\ x)\, S_n(\alpha',\ \beta',\ \gamma',\ y,\ y) =$$

$$= \frac{(\alpha)_m\, (\alpha')_n}{m!\, n!}\, {}_2F_1(-m,\ \beta+\gamma;\ \alpha;\ x)\, {}_2F_1(-n,\ \beta'+\gamma';\ \alpha';\ y) =$$

$$= \Gamma(\alpha+m)\, \Gamma(\alpha'+n) \sum_{k=0}^{m} \sum_{s=0}^{n} \frac{\Gamma(m+n-k-s+1)}{\Gamma(m-k+1)\, \Gamma(n-s+1)}\, \frac{\Gamma(\alpha+\alpha'+k+s-1)}{\Gamma(\alpha+k)\, \Gamma(\alpha'+s)} \cdot$$

$$\cdot \frac{(-1)^{k+s} (\beta+\gamma)_k\, (\beta'+\gamma')_s\, x^k\, y^s}{k!\, s!\, \Gamma(m+n-k-s+1)\, \Gamma(\alpha+\alpha'+k+s-1)}\,.$$

It is known [3]

$$\frac{\Gamma(\mu+v+1)}{\Gamma(\mu+1)\,\Gamma(v+1)}=\frac{2^{\mu+v}}{\pi}\int_{-\pi/2}^{\pi/2}e^{(\mu-v)\theta i}\cos^{\mu+v}\theta\,d\theta \qquad (\mu+v>-1)\,, \tag{13}$$

so that

$$S_m(\alpha,\beta,\gamma,x,x)\;S_n(\alpha',\beta',\gamma',y,y)=$$

$$=\frac{2^{\alpha+\alpha'+m+n-2}}{\pi^2}\,\frac{\Gamma(\alpha+m)\,\Gamma(\alpha'+n)}{\Gamma(\alpha+\alpha'-1)}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}e^{(\alpha-\alpha')\theta i+(m+n)\varphi i}\cdot$$

$$\cdot\cos^{\alpha+\alpha'-2}\theta\,\cos^{m+n}\varphi\sum_{k+s\leq m+n}\frac{(-1)^{k+s}(\beta+\gamma)_k(\beta'+\gamma')_s}{k!\,s!\,(m+n-k-s)!\,(\alpha+\alpha'-1)_{k+s}}\cdot$$

$$\cdot[x\,e^{(\theta-\varphi)i}\cos\theta\sec\varphi]^k\,[y\,e^{-(\theta-\varphi)i}\cos\theta\sec\varphi]^s\,d\theta\,d\varphi\,.$$

Therefore, we arrive at

$$(14)\quad\left\{\begin{aligned}&S_m(\alpha,\beta,\gamma,x,x)\,S_n(\alpha',\beta',\gamma',y,y)=\\&=\frac{2^{\alpha+\alpha'+m+n-2}}{\pi^2}\,\frac{\Gamma(\alpha+m)\,\Gamma(\alpha'+n)}{\Gamma(\alpha+\alpha'+m+n-1)}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}e^{(\alpha-\alpha')\theta i+(m-n)\varphi i}\cos^{\alpha+\alpha'-2}\theta\,\cos^{m+n}\varphi\,\cdot\\&\cdot S_{m+n}(\alpha+\alpha'-1,\beta+\gamma,\beta'+\gamma',xe^{(\theta-\varphi)i}\cos\theta\sec\varphi,ye^{-(\theta-\varphi)i}\cos\theta\sec\varphi)\,d\theta\,d\varphi.\end{aligned}\right.$$

References.

[1] A. Erdélyi, *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York 1953.

[2] L. Weisner, *Group-theoretic origin of certain generating functions*, Pacific J. Math. 5 (1955), 1033-1039.

[3] E. T. Whittaker and G. Watson, *A Course of Modern Analysis*, University Press, Cambridge 1950 (cf. p. 263).

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