Riv. Mat. Univ. Parma (2) 8 (1967), 161-162

S. P. SINGH (*)

A Note «on a Fixed Point Theorem». (**)

Let E be a semi-metric space: that is, a set equipped with a semi-metric d; d is thus a real valued function on $E \times E$ such that

$$d(x, y) = d(y, x) \geqslant 0 , \qquad d(x, x) = 0 , \qquad d(x, y) \leqslant d(x, z) + d(z, y),$$

for arbitrary elements x, y, z of E. [d is a metric if in addition $d(x, y) \neq 0$ for $x \neq y$.] A map U of E into itself is termed a contraction of E if there exists a number k satisfying $0 \leqslant k < 1$ such that

$$d(\mathscr{U}(x), \quad \mathscr{U}(y)) \leqslant k\, d(x, y)$$

for arbitrary x, y in E. Any such contraction of E is obviously a continuous map of E into itself. The following fixed point theorems are well-known.

1. Let (E, d) be a complete semi-metric space, $\mathscr{U}$ a contraction map of E into itself. Then there exists at least one point x of E satisfying $d(\mathscr{U}(x), x) = 0$.

2. Let E be a complete semi-metric space; if S is a non-empty closed subset of E and $\mathscr{U}$: $S \to S$ a contraction, then f has at least one fixed point.

In the present Note we prove the following theorem for semi-metric space.

(*) Indirizzo: Department of Mathematics, University of Windsor, Windsor, Ontario, Canada.

(**) During the writing of this Note the author held a Summer Fellowship of the Canadian Mathematical Congress at Kingston. — Ricevuto: 10-VIII-1967.

Theorem. *Let E be a semi-metric space. Then the following are equivalent:*

(1) *E is complete.*

(2) *If S is any nonempty closed subset of E and $\mathcal{U}: S \to S$ any contraction, then $\mathcal{U}$ has a fixed point.*

Proof.

We prove that (2) implies (1). Suppose that E contains a non-convergent Cauchy sequence $\{x_n\}$. Since $\{x_n\}$ has the Cauchy property, it has no cluster points, i. e. it has no convergent subsequence. Since, moreover, $\{x_n, (n = 0, 1, \ldots)\}$ must be infinite, we may assume that it consists of distinct terms, otherwise, we may select a subsequence which does. We take any $x \in E$, then $l(x) = \inf\{d(x, x_n), x_n = x, (n = 0, 1, \ldots)\} > 0$, because $\{x_n\}$ has no cluster points. We choose any k such that $0 < k < 1$. We define a mapping ϱ of the set of non-negative integers into itself inductively as follows: $\varrho(0) = 0$, and if $n > 1$ and $\varrho(n-1)$ is defined, let $\varrho(n)$ be an integer $> \varrho(n-1)$ such that $d(x_i, x_j) \leqslant r\ l(x_{\varrho(n-1)})$ for all integers $i, j \geqslant \varrho(n)$.

Then $\{x_{\varrho(n)}\}$ is a subsequence of distinct terms and is non-convergent. The set $S = \{x_{\varrho(n)}, (n = 0, 1, \ldots)\}$ is closed and the mapping U: $S \to S$ defined by $\mathcal{U}(x_{\varrho(n)}) = x_{\varrho(n+1)}$ for $n = 0, 1, \ldots,$ is a contraction with no fixed point. The proof is complete.

In case of complete metric space a similar result has been given by Hu [1].

Referencs.

[1] T. K. Hu, *On a fixed point theorem for metric spaces,* Amer. Math. Montly **74** (1967), 436-437.

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