Riv. Mat. Univ. Parma (2) 8 (1967), 163-180

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Formule di derivazione per le funzioni ipergeometriche di Gauss. (**)

1. – La funzione ipergeometrica di GAUSS è definita dalla serie [3]

$$ {}_2F_1(a,\ b;\ c;\ x) = \sum_0^\infty {}_r \frac{(a,\ r)(b,\ r)}{(c,\ r)\ r!}\ x^r , $$

dove c si suppone diverso da zero e da intero negativo, e il simbolo di APPELL $(\lambda,\ r)$, con λ qualsiasi ed r intero positivo o nullo, è definito dalle $(\lambda,\ 0) = 1$, $(\lambda,\ r) = \lambda(\lambda+1)\ldots(\lambda+r-1)$. Per a (oppure b) uguale ad intero negativo si riduce a un polinomio. Così si perviene a quello di JACOBI

$$ P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1,\ n)}{n!}\ {}_2F_1\left(-n,\ \alpha+\beta+n+1;\ \alpha+1;\ \frac{1-x}{2}\right), $$

dal quale si possono dedurre i polinomi *ultrasferici* e di LEGENDRE, di LAGUERRE, di HERMITE.

Quale serie richiede la condizione di convergenza $|x| < 1$, e comprende un insieme notevole di *funzioni speciali*.

Sulla funzione di GAUSS sono note le seguenti fondamentali formule di deri-

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(**) Ricevuto: 26-X-1966.

vazione che riportiamo dall'opera [2] e dalla Nota [1] (¹), con la posizione $\frac{\mathrm{d}}{\mathrm{d}x} = \mathrm{D}_x = \mathrm{D}$:

$$\text{(1)} \qquad \mathrm{D}^r[x^{a+r-1}\,{}_2F_1(a,\ b;\ c;\ x)] = (a,\ r)\,x^{a-1}\,{}_2F_1(a+r,\ b;\ c;\ x),$$

$$\text{(2)} \qquad \mathrm{D}^r[x^{c-1}\,{}_2F_1(a,\ b;\ c;\ x)] = (c-r,\ r)\,x^{c-r-1}\,{}_2F_1(a,\ b;\ c-r;\ x),$$

$$\text{(3)} \qquad \mathrm{D}^r\,{}_2F_1(a,\ b;\ c;\ x) = \frac{(a,\ r)(b,\ r)}{(c,\ r)}\,{}_2F_1(a+r,\ b+r;\ c+r;\ x),$$

$$\text{(4)} \qquad \mathrm{D}^r[x^{c-a+r-1}\,(1-x)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ x)] = \\ = (c-a,\ r)\,x^{c-a-1}\,(1-x)^{a+b-c-r}\,{}_2F_1(a-r,\ b;\ c;\ x),$$

$$\text{(5)} \qquad \mathrm{D}^r[x^{c-1}\,(1-x)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ x)] = \\ = (c-r,\ r)\,x^{c-r-1}\,(1-x)^{a+b-c-r}\,{}_2F_1(a-r,\ b-r;\ c-r;\ x),$$

$$\text{(6)} \qquad \mathrm{D}^r[(1-x)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ x)] = \\ = \frac{(c-a,\ r)(c-b,\ r)}{(c,\ r)}\,(1-x)^{a+b-c-r}\,{}_2F_1(a,\ b;\ c+r;\ x)\,,$$

$$\text{(7)} \qquad \mathrm{D}^r[(1-x)^{a+r-1}\,{}_2F_1(a,\ b;\ c;\ x)] = \\ = (-1)^r\,\frac{(a,\ r)(c-b,\ r)}{(c,\ r)}\,(1-x)^{a-1}\,{}_2F_1(a+r,\ b;\ c+r;\ x),$$

$$\text{(8)} \qquad \mathrm{D}^r[x^{c-1}(1-x)^{b-c+r}\,{}_2F_1(a,\ b;\ c;\ x)] = \\ = (c-r,\ r)\,x^{c-r-1}\,(1-x)^{b-c}\,{}_2F_1(a-r,\ b;\ c-r;\ x).$$

Da esse prende ora le mosse il nostro lavoro di ricerca sistematica di formule di derivazione, rispetto alla variabile x, di funzioni di Gauss ad argomento $1/x$ o x^2. E verremo a stabilire un notevole e completo gruppo di formule di derivazione (delle quali solo qualcuna risulta nota in generale o in particolare), tutte utili nella teoria delle *funzioni speciali*.

La ricerca fa seguito ad altra nostra [7] sulle funzioni ipergeometriche di Kummer

$${}_1F_1(a;\ c;\ x) = \sum_0^\infty{}_r\,\frac{(a,\ r)}{(c,\ r)}\,\frac{x^r}{r!}\,.$$

(¹) Cfr. l'elenco dei « Lavori consultati », alla fine di questo lavoro.

2. – Premettiamo che

$$-x^{-2}\,\mathrm{D}_{1/x}=\mathrm{D}_x\,, \qquad x^2\,\mathrm{D}_x=-\mathrm{D}_{1/x}\,,$$

e che

$$\underleftarrow{(x^2\,\mathrm{D}_x)^r}=\underleftarrow{x^{r+1}\,\mathrm{D}_x^r\,x^{r-1}}\;,$$

per cui

$$\mathrm{D}_{1/x}^r=(-1)^r\,\underleftarrow{x^{r+1}\,\mathrm{D}_x^r\,x^{r-1}}\;.$$

Allora dalle formule fondamentali (1), ..., (8) seguono le seguenti:

(9) $$x^r\,\mathrm{D}^r\left[x^{-a}\,{}_2F_1(a,\ b;\ c;\ 1/x)\right]=(-1)^r\,(a,\ r)\,x^{-a}\,{}_2F_1(a+r,\ b;\ c;\ 1/x),$$

(10) $$\underleftarrow{\mathrm{D}^r\,x^r}\left[x^{-c}\,{}_2F_1(a,\ b;\ c;\ 1/x)\right]=(1-c,\ r)\,x^{-c}\,{}_2F_1(a,\ b;\ c-r;\ 1/x),$$

(11) $$\underleftarrow{\mathrm{D}^r\,x^r}\left[x^{-1}\,{}_2F_1(a,\ b;\ c;\ 1/x)\right]=$$

$$=(-1)^r\,\frac{(a,\ r)(b,\ r)}{(c,\ r)}\,x^{-r-1}\,{}_2F_1(a+r,\ b+r;\ c+r;\ 1/x),$$

(12) $$\mathrm{D}^r\left[x^{-b}\,(x-1)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ 1/x)\right]=$$

$$=(-1)^r\,(c-a,\ r)\,x^{-b}\,(x-1)^{a+b-c-r}\,{}_2F_1(a-r,\ b;\ c;\ 1/x),$$

(13) $$\underleftarrow{\mathrm{D}^r\,x^r}\left[x^{-a-b}\,(x-1)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ 1/x)\right]=$$

$$=(1-c,\ r)\,x^{-a-b+r}\,(x-1)^{a+b-c-r}\,{}_2F_1(a-r,\ b-r;\ c-r;\ 1/x),$$

(14) $$\underleftarrow{\mathrm{D}^r\,x^r}\left[x^{-a-b+c-1}\,(x-1)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ 1/x)\right]=$$

$$=(-1)^r\,\frac{(c-a,\ r)(c-b,\ r)}{(c,\ r)}\,x^{-a-b+c-1}(x-1)^{a+b-c-r}\,{}_2F_1(a,\ b;\ c+r;\ 1/x),$$

(15) $$\mathrm{D}^r\left[x^{-a}\,(x-1)^{a+r-1}\,{}_2F_1(a,\ b;\ c;\ 1/x)\right]=$$

$$=\frac{(a,\ r)(c-b,\ r)}{(c,\ r)}\,x^{-a-r}\,(x-1)^{a-1}\,{}_2F_1(a+r,\ b;\ c+r;\ 1/x),$$

(16) $$\mathrm{D}^r\left[x^{-b}\,(x-1)^{b-c+r}\,{}_2F_1(a,\ b;\ c;\ 1/x)\right]=$$

$$=(1-c,\ r)\,x^{-b}\,(x-1)^{b-c}\,{}_2F_1(a-r,\ b;\ c-r;\ 1/x)\,.$$

Le (12), (15), (16), sostituendo x con $x+1$, si possono presentare nell'altra forma:

(12′)
$$x^r\,\mathrm{D}^r\left[x^{a+b-c}\,(x+1)^{-b}\,{}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)\right]=$$
$$=(-1)^r\,(c-a,\ r)\,x^{a+b-c}\,(x+1)^{-b}\,{}_2F_1\big(a-r,\ b;\ c;\ 1/(x+1)\big),$$

(15′)
$$\underleftarrow{\mathrm{D}^r\,x^r}\left[x^{a-1}\,(x+1)^{-a}\,{}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)\right]=$$
$$=\frac{(a,\ r)(c-b,\ r)}{(c,\ r)}\,x^{a-1}\,(x+1)^{-a-r}\,{}_2F_1\big(a+r,\ b;\ c+r;\ 1/(x+1)\big),$$

(16′)
$$\underleftarrow{\mathrm{D}^r\,x^r}\left[x^{b-c}\,(x+1)^{-b}\,{}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)\right]=$$
$$=(1-c,\ r)\,x^{b-c}\,(x+1)^{-b}\,{}_2F_1\big(a-r,\ b;\ c-r;\ 1/(x+1)\big).$$

3. – Richiamiamo ora gli sviluppi su operatori differenziali [6]:

$$(-1)^{m-1}\,\underleftarrow{(\mathrm{D}\,x)}^m=\sum_1^m{}_r\,(-1)^{r-1}\,K_{m,r}\,\underleftarrow{\mathrm{D}^r\,x^r}\,,$$

$$(-1)^m\,\underleftarrow{(x\,\mathrm{D})}^m=\sum_0^m{}_r\,(-1)^r\,K_{m+1,\ r+1}\,\underleftarrow{\mathrm{D}^r\,x^r}\,,$$

$$\underleftarrow{(x\,\mathrm{D})}^m=\sum_1^m{}_r\,K_{m,r}\,x^r\,\mathrm{D}^r\,,$$

$$\underleftarrow{(\mathrm{D}\,x)}^m=\sum_0^m{}_r\,K_{m+1,r+1}\,x^r\,\mathrm{D}^r\,,$$

nei cui coefficienti figurano i numeri di STIRLING di seconda specie definiti dalle posizioni

$$K_{m,1}=K_{m,m}=1\,,$$

$$K_{m,r}=K_{m-1,\ r-1}+rK_{m-1,r}\qquad\text{per }\ 1<r<m,$$

$$K_{m,0}=0\,,\qquad K_{m,r}=0\,,\qquad\text{per }\ r>m.$$

Associando a questi sviluppi le precedenti formule (8), (9), (10), (11), (12′), (13), (14), (15′), (16′), si deducono le altre formule seguenti:

(17)
$$\underleftarrow{(x\,\mathrm{D})}^m \left[x^{-a}\, {}_2F_1(a,\ b;\ c;\ 1/x)\right] =$$
$$= \sum_1^m {}_r\, (-1)^r\, K_{m,r}\,(a,\ r)\, x^{-a}\, {}_2F_1\,(a+r,\ b;\ c;\ 1/x),$$

(18)
$$\underleftarrow{(\mathrm{D}\,x)}^m \left[x^{-a}\, {}_2F_1(a,\ b;\ c;\ 1/x)\right] =$$
$$= \sum_0^m {}_r\, (-1)^r\, K_{m+1,\ r+1}(a,\ r)\, x^{-a}\, {}_2F_1(a+r,\ b;\ c;\ 1/x),$$

(19)
$$(-1)^{m-1}\, \underleftarrow{(\mathrm{D}\,x)}^m \left[x^{-c}\, {}_2F_1(a,\ b;\ c;\ 1/x)\right] =$$
$$= \sum_1^m {}_r\, (-1)^{r-1}\, K_{m,r}\,(1-c,\ r)\, x^{-c}\, {}_2F_1(a,\ b;\ c-r;\ 1/x),$$

(20)
$$(-1)^m\, \underleftarrow{(x\,\mathrm{D})}^m \left[x^{-c}\, {}_2F_1(a,\ b;\ c;\ 1/x)\right] =$$
$$= \sum_0^m {}_r\, (-1)^r\, K_{m+1,\ r+1}\,(1-c,\ r)\, x^{-c}\, {}_2F_1(a,\ b;\ c-r;\ 1/x),$$

(21)
$$(-1)^m\, \underleftarrow{(\mathrm{D}\,x)}^m \left[x^{-1}\, {}_2F_1(a,\ b;\ c;\ 1/x)\right] =$$
$$= \sum_1^m {}_r\, K_{m,r}\, \frac{(a,\ r)(b,\ r)}{(c,\ r)}\, x^{-r-1}\, {}_2F_1(a+r,\ b+r;\ c+r;\ 1/x),$$

(22)
$$(-1)^m\, \underleftarrow{(x\,\mathrm{D})}^m \left[x^{-1}\, {}_2F_1(a,\ b;\ c;\ 1/x)\right] =$$
$$= \sum_0^m {}_r\, K_{m+1,\ r+1}\, \frac{(a,\ r)(b,\ r)}{(c,\ r)}\, x^{-r-1}\, {}_2F_1(a+r,\ b+r;\ c+r;\ 1/x),$$

(23)
$$\underleftarrow{(x\,\mathrm{D})}^m \left[x^{a+b-c}\,(x+1)^{-b}\, {}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)\right] =$$
$$= x^{a+b-c}\,(x+1)^{-b} \sum_1^m {}_r\, (-1)^r\, K_{m,r}\,(c-a,\ r)\, {}_2F_1\big(a-r,\ b;\ c;\ 1/(x+1)\big),$$

(24)
$$\underleftarrow{(\mathrm{D}\,x)}^m \left[x^{a+b-c}\,(x+1)^{-b}\, {}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)\right] =$$
$$= x^{a+b-c}\,(x+1)^{-b} \sum_0^m {}_r\, (-1)^r\, K_{m+1,\ r+1}\,(c-a,\ r)\ \cdot$$
$$\cdot\ {}_2F_1\big(a-r,\ b;\ c;\ 1/(x+1)\big),$$

$$(25)\qquad (-1)^{m-1}\,\underleftarrow{(\mathrm{D}\,x)^m}\,[x^{-a-b}\,(x-1)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ 1/x)] =$$

$$= x^{-a-b}\,(x-1)^{a+b-c}\sum_{1}^{m}{}_r\,(-1)^{r-1}\,K_{m,r}\,(1-c,\,r)\,\big(x/(x-1)\big)^r\cdot$$

$$\cdot\,{}_2F_1(a-r,\,b-r;\,c-r;\,1/x),$$

$$(26)\qquad (-1)^{m}\,\underleftarrow{(x\,\mathrm{D})^m}\,[x^{-a-b}\,(x-1)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ 1/x)] =$$

$$= x^{-a-b}\,(x-1)^{a+b-c}\sum_{0}^{m}{}_r\,(-1)^{r}\,K_{m+1,\,r+1}\,(1-c,\,r)\,\big(x/(x-1)\big)^r\cdot$$

$$\cdot\,{}_2F_1(a-r,\ b-r;\ c-r;\ 1/x),$$

$$(27)\qquad (-1)^{m}\,\underleftarrow{(\mathrm{D}\,x)^m}\,[x^{-a-b+c-1}\,(x-1)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ 1/x)] = x^{-a-b+c-1}\cdot$$

$$\cdot\,(x-1)^{a+b-c}\sum_{1}^{m}{}_r\,K_{m,r}\,\frac{(c-a,\,r)(c-b,\,r)}{(c,\ r)}\,(x-1)^{-r}\,{}_2F_1(a,\ b;\ c+r;\ 1/x),$$

$$(28)\qquad (-1)^{m}\,\underleftarrow{(x\,\mathrm{D})^m}\,[x^{-a-b+c-1}\,(x-1)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ 1/x)] = x^{-a-b+c-1}\cdot$$

$$\cdot\,(x-1)^{a+b-c}\sum_{0}^{m}{}_r\,K_{m+1,\ r+1}\,\frac{(c-a,\ r)(c-b,\ r)}{(c,\ r)}\,(x-1)^{-r}\,{}_2F_1(a,\ b;\ c+r;\ 1/x),$$

$$(29)\qquad (-1)^{m-1}\,\underleftarrow{(\mathrm{D}\,x)^m}\,[x^{a-1}\,(x+1)^{-a}\,{}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)] =$$

$$= x^{a-1}\,(x+1)^{-a}\sum_{1}^{m}{}_r\,(-1)^{r-1}K_{m,r}\,\frac{(a,\ r)(c-b,\ r)}{(c,\ r)}\,(x+1)^{-r}\cdot$$

$$\cdot\,{}_2F_1\big(a+r,\ b;\ c+r;\ 1/(x+1)\big),$$

$$(30)\qquad (-1)^{m}\,\underleftarrow{(x\,\mathrm{D})^m}\,[x^{a-1}\,(x+1)^{-a}\,{}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)] =$$

$$= x^{a-1}\,(x+1)^{-a}\sum_{0}^{m}{}_r\,(-1)^{r}\,K_{m+1,\,r+1}\,\frac{(a,\ r)(c-b,\ r)}{(c,\ r)}\,(x+1)^{-r}\cdot$$

$$\cdot\,{}_2F_1\big(a+r,\ b;\ c+r;\ 1/(x+1)\big),$$

$$(31)\qquad (-1)^{m-1}\,\underleftarrow{(\mathrm{D}\,x)^m}\,[x^{b-c}\,(x+1)^{-b}\,{}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)] = x^{b-c}\,(x+1)^{-b}\cdot$$

$$\cdot\sum_{1}^{m}{}_r\,(-1)^{r-1}\,K_{m,r}\,(1-c,\ r)\,{}_2F_1\big(a-r,\ b;\ c-r;\ 1/(x+1)\big),$$

$$(32)\qquad (-1)^{m}\,\underleftarrow{(x\,\mathrm{D})^m}\,[x^{b-c}\,(x+1)^{-b}\,{}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)] = x^{b-c}\,(x+1)^{-b}\cdot$$

$$\cdot\sum_{0}^{m}{}_r\,(-1)^{r}\,K_{m+1,\ r+1}\,(1-c,\ r)\,{}_2F_1\big(a-r,\ b;\ c-r;\ 1/(x+1)\big).$$

D'altra parte è

$$\underleftarrow{x^c\,(x\,\mathrm{D})^m\,x^{-c}} = \underleftarrow{(x\,\mathrm{D} - c)^m}\,,$$

ed ancora

$$\underleftarrow{x^c\,(\mathrm{D}\,x)^m\,x^{-c}} = \underleftarrow{x^{c-1}\,(x\,\mathrm{D})^m\,x^{1-c}} = x^{-1}\,\underleftarrow{x^c\,(x\,\mathrm{D})^m\,x^{-c}}\;x = x^{-1}\,\underleftarrow{(x\,\mathrm{D} - c)^m}\,x\,.$$

Per queste le (17), ..., (32) assumono la forma elegante e notevole seguente:

(17′) $$\underleftarrow{(x\,\mathrm{D} - a)^m}\;{}_2F_1(a,\ b;\ c;\ 1/x) = \sum_1^m{}_r\,(-1)^r\,K_{m,r}\,(a,\ r)\;{}_2F_1(a + r,\ b;\ c;\ 1/x),$$

(18′) $$\underleftarrow{(\mathrm{D}\,x - a)^m}\;{}_2F_1(a,\ b;\ c;\ 1/x) = \sum_0^m{}_r\,(-1)^r\,K_{m+1,r+1}\,(a,\ r)\;{}_2F_1(a + r,\ b;\ c;\ 1/x),$$

(19′) $$\underleftarrow{(c - \mathrm{D}\,x)^m}\;{}_2F_1(a,\ b;\ c;\ 1/x) = \sum_1^m{}_r\,(-1)^r\,K_{m,r}\,(1 - c,\ r)\;{}_2F_1(a,\ b;\ c - r;\ 1/x),$$

(20′) $$\underleftarrow{(c - x\,\mathrm{D})^m}\;{}_2F_1(a,\ b;\ c;\ 1/x) = \sum_0^m{}_r\,(-1)^r\,K_{m+1,\,r+1}\,(1 - c,\ r)\;{}_2F_1(a,\ b;\ c - r;\ 1/x),$$

(21′) $$(-1)^m\,\underleftarrow{(x\,\mathrm{D})^m}\;{}_2F_1(a,\ b;\ c;\ 1/x) = \sum_1^m{}_r\,K_{m,r}\,\frac{(a,\ r)(b,\ r)}{(c,\ r)}\,x^{-r}\;{}_2F_1(a + r,\ b + r;\ c + r;\ 1/x),$$

(22′) $$\underleftarrow{(1 - x\,\mathrm{D})^m}\;{}_2F_1(a,\ b;\ c;\ 1/x) = \sum_0^m{}_r\,K_{m+1,r+1}\,\frac{(a,\ r)(b,\ r)}{(c,\ r)}\,x^{-r}\;{}_2F_1(a + r,\ b + r;\ c + r;\ 1/x),$$

(23′) $$\underleftarrow{(x\mathrm{D} + a + b - c)^m}\,[(x + 1)^{-b}\;{}_2F_1(a,\ b;\ c;\ 1/(x + 1))] = (x + 1)^{-b}\sum_1^m{}_r\,(-1)^r\,K_{m,r}(c - a,\ r)\;{}_2F_1\,(a - r,\ b;\ c;\ 1/(x + 1)),$$

(24′) $$\underleftarrow{(\mathrm{D}\,x + a + b - c)^m}\,[(x + 1)^{-b}\;{}_2F_1(a,\ b;\ c;\ 1/(x + 1))] = (x + 1)^{-b}\sum_0^m{}_r\,(-1)^r\,K_{m+1,r+1}\,(c - a,\ r)\;{}_2F_1(a - r,\ b;\ c;\ 1/(x + 1)),$$

(25′) $$\underleftarrow{(a+b-\mathrm{D}\,x)}^m\,[(x-1)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ 1/x)]=$$

$$=(x-1)^{a+b-c}\sum_{1}^{m}{}_r\,(-1)^r K_{m,r}(1-c,\ r)\left(\frac{x}{x-1}\right)^r {}_2F_1(a-r,\ b-r;\ c-r;\ 1/x),$$

(26′) $$\underleftarrow{(a+b-x\,\mathrm{D})}^m\,[(x-1)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ 1/x)]=$$

$$=(x-1)^{a+b-c}\sum_{0}^{m}{}_r\,(-1)^r K_{m+1,\,r+1}(1-c,r)\left(\frac{x}{x-1}\right)^r {}_2F_1(a-r,\ b-r;c-r;1/x),$$

(27′) $$\underleftarrow{(a+b-c+1-\mathrm{D}\,x)}^m\,[(x-1)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ 1/x)]=$$

$$=(x-1)^{a+b-c}\sum_{1}^{m}{}_r\,K_{m,r}\frac{(c-a,\ r)(c-b,\ r)}{(c,\ r)}(x-1)^{-r}\,{}_2F_1(a,\ b;\ c+r;\ 1/x),$$

(28′) $$\underleftarrow{(a+b-c+1-x\,\mathrm{D})}^m\,[(x-1)^{a+b-c}\,{}_2F_1(a,\ b;\ c;\ 1/x)]=$$

$$=(x-1)^{a+b-c}\sum_{0}^{m}{}_r\,K_{m+1,r+1}\frac{(c-a,\ r)(c-b,\ r)}{(c,\ r)}(x-1)^{-r}\,{}_2F_1(a,b;\ c+r;\ 1/x),$$

(29′) $$\underleftarrow{(1-a-\mathrm{D}\,x)}^m\,[(x+1)^{-a}\,{}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)]=$$

$$=(x+1)^{-a}\sum_{1}^{m}{}_r\,(-1)^r K_{m,r}\frac{(a,\ r)(c-b,\ r)}{(c,\ r)}(x+1)^{-r}\,{}_2F_1\big(a+r,b;c+r;1/(x+1)\big),$$

(30′) $$\underleftarrow{(1-a-x\,\mathrm{D})}^m\,[(x+1)^{-a}\,{}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)]=$$

$$=(x+1)^{-a}\sum_{0}^{m}{}_r(-1)^r K_{m+1,r+1}\frac{(a,\ r)(c-b,\ r)}{(c,\ r)}(x+1)^{-r}\,{}_2F_1\big(a+r,b;c+r;1/(x+1)\big),$$

(31′) $$\underleftarrow{(c-b-\mathrm{D}\,x)}^m\,[(x+1)^{-b}\,{}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)]=$$

$$=(x+1)^{-b}\sum_{1}^{m}{}_r\,(-1)^r K_{m,r}\,(1-c,\ r)\,{}_2F_1\big(a-r,\ b;\ c-r;\ 1/(x+1)\big),$$

(32′) $$\underleftarrow{(c-b-x\,\mathrm{D})}^m\,[(x+1)^{-b}\,{}_2F_1\big(a,\ b;\ c;\ 1/(x+1)\big)]=$$

$$=(x+1)^{-b}\sum_{0}^{m}{}_r\,(-1)^r K_{m+1,r+1}\,(1-c,\ r)\,{}_2F_1\big(a-r,\ b;c-r;1/(x+1)\big).$$

4. – Tutte le formule stabilite si possono specializzare per le funzioni di KUMMER. Posto $b\,x$ in luogo di x, osservando che $\mathrm{D}_{bx}^{r} = b^{-r}\,\mathrm{D}_{x}^{r}$, passiamo al limite per $b \to \infty$, tenendo presente che

$$\lim_{b\to\infty} \frac{(b,\ r)}{b^r} = 1, \qquad \lim_{b\to\infty} \frac{(c-b,\ r)}{b^r} = (-1)^r, \qquad \lim_{b\to\infty}\left(1-\frac{1}{bx}\right)^b = e^{-1/x}.$$

Si ritrovano le formule [7]:

$$x^r\,\mathrm{D}^r\,[x^{-a}\,{}_1F_1(a;\ c;\ 1/x)] = (-1)^r\,(a,\ r)\,x^{-a}\,{}_1F_1(a+r;\ c;\ 1/x),$$

$$\underleftarrow{\mathrm{D}^r\,x^r}\,[x^{-c}\,{}_1F_1(a;\ c;\ 1/x)] = (1-c,\ r)\,x^{-c}\,{}_1F_1\,(a;\ c-r;\ 1/x),$$

$$\underleftarrow{\mathrm{D}^r\,x^r}\,[x^{-1}\,{}_1F_1(a;\ c;\ 1/x)] = (-1)^r\,\frac{(a,\ r)}{(c,\ r)}\,x^{-r-1}\,{}_1F_1(a+r;\ c+r,\ 1/x)\,,$$

$$x^r\,\mathrm{D}^r\,[x^{a-c}\,e^{-1/x}\,{}_1F_1(a;\,c;\,1/x)] = (-1)^r\,(c-a,\,r)\,x^{a-c}\,e^{-1/x}\,{}_1F_1(a-r;\,c;\,1/x),$$

$$\underleftarrow{\mathrm{D}^r\,x^r}\,[x^{-c}\,e^{-1/x}\,{}_1F_1(a;\ c;\ 1/x)] = (1-c,\,r)\,x^{-c}\,e^{-1/x}\,{}_1F_1(a-r;\ c-r;\ 1/x),$$

$$\underleftarrow{\mathrm{D}^r\,x^r}\,[x^{-1}\,e^{-1/x}\,{}_1F_1(a;\,c;\,1/x)] = \frac{(c-a,\ r)}{(c,\ r)}\,x^{-r-1}\,e^{-1/x}\,{}_1F_1(a;\ c+r;\ 1/x)\,.$$

E si ottengono le altre formule:

$$\underleftarrow{(x\,\mathrm{D}-a)^m}\,{}_1F_1(a;\ c;\ 1/x) = \sum_{1}^{m}{}_r\,(-1)^r\,K_{m,r}\,(a,\ r)\,{}_1F_1(a+r;\ c;\ 1/x),$$

$$\underleftarrow{(\mathrm{D}\,x-a)^m}\,{}_1F_1(a;\ c;\ 1/x) = \sum_{0}^{m}{}_r\,(-1)^r\,K_{m+1,\ r+1}\,(a,\ r)\,{}_1F_1(a+r;\ c;\ 1/x),$$

$$\underleftarrow{(c-\mathrm{D}\,x)^m}\,{}_1F_1(a;\ c;\ 1/x) = \sum_{1}^{m}{}_r\,(-1)^r\,K_{m,r}\,(1-c,\ r)\,{}_1F_1(a;\ c-r;\ 1/x),$$

$$\underleftarrow{(c-x\,\mathrm{D})^m}\,{}_1F_1(a;\ c;\ 1/x) = \sum_{0}^{m}{}_r\,(-1)^r K_{m+1,\ r+1}\,(1-c,\ r)\,{}_1F_1\,(a;\ c-r;\ 1/x),$$

$$(-1)^m\,\underleftarrow{(x\,\mathrm{D})^m}\,{}_1F_1(a;\ c;\ 1/x) = \sum_{1}^{m}{}_r\,K_{m,r}\,\frac{(a,\ r)}{(c,\ r)}\,x^{-r}\,{}_1F_1(a+r;\ c+r;\ 1/x),$$

$$\underleftarrow{(1-x\,\mathrm{D})^m}\,{}_1F_1(a;\ c;\ 1/x) = \sum_{0}^{m}{}_r\,K_{m+1,\ r+1}\,\frac{(a,\ r)}{(c,\ r)}\,x^{-r}\,{}_1F_1(a+r;\ c+r;\ 1/x),$$

$$\underleftarrow{(x\,\mathrm{D}+a-c)^m}\,[e^{-1/x}\,{}_1F_1(a;\ c;\ 1/x)] =$$

$$= e^{-1/x}\sum_{1}^{m}{}_r\,(-1)^r\,K_{m,r}\,(c-a,\ r)\,{}_1F_1(a-r;\ c;\ 1/x),$$

$$\underleftarrow{(\mathrm{D}\, x + a - c)}^m \left[e^{-1/x}\, {}_1F_1(a;\; c;\; 1/x)\right] =$$

$$= e^{-1/x} \sum_0^m {}_r\, (-1)^r\, K_{m+1,\, r+1}\, (c - a,\; r)\, {}_1F_1(a - r;\; c;\; 1/x),$$

$$\underleftarrow{(c - \mathrm{D}\, x)}^m \left[e^{-1/x}\, {}_1F_1(a;\; c;\; 1/x)\right] =$$

$$= e^{-1/x} \sum_1^m {}_r\, (-1)^r\, K_{m,r}\, (1 - c,\; r)\, {}_1F_1(a - r;\; c - r;\; 1/x),$$

$$\underleftarrow{(c - x\, \mathrm{D})}^m \left[e^{-1/x}\, {}_1F_1(a;\; c;\; 1/x)\right] =$$

$$= e^{-1/x} \sum_0^m {}_r\, (-1)^r\, K_{m+1,\, r+1}\, (1 - c,\; r)\, {}_1F_1(a - r;\; c - r;\; 1/x),$$

$$(-1)^{m-1}\, \underleftarrow{(x\, \mathrm{D})}^m \left[e^{-1/x}\, {}_1F_1(a;\; c;\; 1/x)\right] =$$

$$= e^{-1/x} \sum_1^m {}_r\, (-1)^{r-1} K_{m,\, r}\, \frac{(c - a,\; r)}{(c,\; r)}\, x^{-r}\, {}_1F_1(a;\; c + r;\; 1/x),$$

$$\underleftarrow{(1 - x\, \mathrm{D})}^m \left[e^{-1/x}\, {}_1F_1(a;\; c;\; 1/x)\right] =$$

$$= e^{-1/x} \sum_0^m {}_r\, (-1)^r\, K_{m+1,\, r+1}\, \frac{(c - a,\; r)}{(c,\; r)}\, x^{-r}\, {}_1F_1(a;\; c + r;\; 1/x).$$

Dalla (9) per a uguale all'intero negativo $-n$ si ha

$$\mathrm{D}^r[x^n\, {}_2F_1(-n,\; b;\; c;\; 1/x)] = \binom{n}{r}\, r!\;\; x^{n-r}\, {}_2F_1(-n + r,\; b;\; c;\; 1/x),$$

e si ritrova che i reciproci dei polinomi ipergeometrici ${}_2F_1(-n,\; b;\; c;\; x)$ appartengono alla classe di APPELL.

Se si fa $a = -n$ nella (12) si ha la formula notevole

$$ {}_2F_1(-n - r,\; b;\; c;\; 1/x) = \frac{(-1)^r}{(c + n,\; r)}\, x^b\, (x - 1)^{c-b+n+r} \cdot$$

$$\cdot \mathrm{D}^r \left[x^{-b}\, (x - 1)^{b-c-n}\, {}_2F_1(-n,\; b;\; c;\; 1/x)\right],$$

dalla quale si deduce con operazione limite

$$ {}_1F_1(-n - r;\; c;\; 1/x) = \frac{(-1)^r}{(c + n,\; r)}\, x^{c+n+r}\, e^{1/x}\, \mathrm{D}^r[x^{-c-n}\, e^{-1/x}\, {}_1F_1(-n;\; c;\; 1/x)].$$

E questa, introducendo i polinomi di LAGUERRE

$$L_n^{(c)}(x) = \frac{(c+1,\ n)}{n\,!}\ {}_1F_1(-n;\ c+1;\ x),$$

si può scrivere nella forma

$$L_{n+r}^{(c-1)}(1/x) = (-1)^r \frac{n\,!}{(n+r)\,!}\ x^{c+n+r}\ e^{1/x}\ \mathrm{D}^r[x^{-c-n}\ e^{-1/x}\ L_n^{(c-1)}(1/x)].$$

5. – Passiamo alla seconda parte del lavoro e ricerchiamo formule di derivazione di funzioni di GAUSS ad argomento x^2. Esse seguono dalle formule su operatori differenziali [4]:

$$\mathrm{D}_x^{2m} = \underleftarrow{2^{2m}\, x\, \mathrm{D}_{x^2}^m\, x^{2m-1}\, \mathrm{D}_{x^2}^m}, \qquad \mathrm{D}_x^{2m} = \underleftarrow{2^{2m}\, \mathrm{D}_{x^2}^m\, x^{2m+1}\, \mathrm{D}_{x^2}^m\, x^{-1}},$$

$$\mathrm{D}_x^{2m+1} = \underleftarrow{2^{2m+1}\, \mathrm{D}_{x^2}^m\, x^{2m+1}\, \mathrm{D}_{x^2}^{m+1}}, \qquad \mathrm{D}_x^{2m+1} = \underleftarrow{2^{2m+1}\, x\, \mathrm{D}_{x^2}^{m+1}\, x^{2m+1}\, \mathrm{D}_{x^2}^m\, x^{-1}},$$

con procedimento che non riportiamo ma che può essere facilmente ricostruito. Otteniamo:

(33) $$\mathrm{D}^{2m}[x^{2c-1}\ {}_2F_1(c-m+\tfrac{1}{2},\ b;\ c;\ x^2)] =$$
$$= (1-2c,\ 2m)\ x^{2c-2m-1}\ {}_2F_1(c+\tfrac{1}{2},\ b;\ c-m;\ x^2),$$

(34) $$\mathrm{D}^{2m}[x\ {}_2F_1(-m+3\tfrac{1}{2},\ b;\ c;\ x^2)] =$$
$$= (-1)^m\, 2^{2m}\, (-\tfrac{1}{2},\ m)(3\tfrac{1}{2},\ m)\, \frac{(b,\ m)}{(c,\ m)}\, x\ {}_2F_1(m+3\tfrac{1}{2},\ b+m;\ c+m;\ x^2),$$

(35) $$\mathrm{D}^{2m}[x^{2c-2}\ {}_2F_1(c-m-\tfrac{1}{2},\ b;\ c;\ x^2)] =$$
$$= (2-2c,\ 2m)\ x^{2c-2m-2}\ {}_2F_1(c-\tfrac{1}{2},\ b;\ c-m;\ x^2),$$

(36) $$\mathrm{D}^{2m}\ {}_2F_1(-m+\tfrac{1}{2},\ b;\ c;\ x^2) =$$
$$= (-1)^m\, 2^{2m}\, (\tfrac{1}{2},\ m)^2 \big| \frac{(b,\ m)}{(c,\ m)}\ {}_2F_1(m+\tfrac{1}{2},\ b+m;\ c+m;\ x^2),$$

(37) $$\mathrm{D}^{2m+1}[x^{2c-1}\ {}_2F_1(c-m-\tfrac{1}{2},\ b;\ c;\ x^2)] =$$
$$= -(1-2c,\ 2m+1)\ x^{2c-2m-2}\ {}_2F_1(c+\tfrac{1}{2},\ b;\ c-m;\ x^2),$$

(38) $$D^{2m+1}[x\ {}_2F_1(-m+\tfrac{1}{2},\ b;\ c;\ x^2)] =$$

$$= (-1)^m 2^{2m}\ (\tfrac{1}{2},\ m)(3\tfrac{1}{2},\ m)\ \frac{(b,\ m)}{(c,\ m)}\ {}_2F_1(m+3\tfrac{1}{2},\ b+m;\ c+m;\ x^2),$$

(39) $$D^{2m+1}[x^{2c-2}\ {}_2F_1(c-m-\tfrac{1}{2},\ b;\ c;\ x^2)] =$$

$$= -(2-2c,\ 2m+1)\ x^{2c-2m-3}\ {}_2F_1(c-\tfrac{1}{2},\ b;\ c-m-1;\ x^2),$$

(40) $$D^{2m+1}\ {}_2F_1(-m+\tfrac{1}{2},\ b;\ c;\ x^2) =$$

$$=(-1)^m 2^{2m}(\tfrac{1}{2},\ m)(3\tfrac{1}{2},\ m)\ \frac{(b,m+1)}{(c,m+1)}\ x\ {}_2F_1(m+3\tfrac{1}{2}, b+m+1; c+m+1; x^2),$$

(41) $$D^{2m}[x\ {}_2F_1(a,\ b;\ 3\tfrac{1}{2};\ x^2)] =$$

$$= 2^{2m}\ (a,\ m)(b,\ m)\ x\ {}_2F_1(a+m,\ b+m;\ 3\tfrac{1}{2};\ x^2),$$

(42) $$D^{2m}\ {}_2F_1(a,\ b;\ \tfrac{1}{2};\ x^2) = 2^{2m}(a,\ m)(b,\ m)\ {}_2F_1(a+m,\ b+m;\ \tfrac{1}{2};\ x^2),$$

(43) $$D^{2m+1}[x\ {}_2F_1(a,\ b;\ 3\tfrac{1}{2};\ x^2)] = 2^{2m}(a,\ m)(b,\ m)\ {}_2F_1(a+m,\ b+m;\ \tfrac{1}{2};\ x^2),$$

(44) $$D^{2m+1}\ {}_2F_1(a,\ b;\ \tfrac{1}{2};\ x^2) =$$

$$= 2^{2m+2}\ (a,\ m+1)(b,\ m+1)\ x\ {}_2F_1(a+m+1,\ b+m+1;\ 3\tfrac{1}{2};\ x^2),$$

(45) $$D^{2m}[x^{2c-1}\ (1-x^2)^{b-c+m-1/2}\ {}_2F_1(m-\tfrac{1}{2},\ b;\ c;\ x^2)] =$$

$$= (1-2c,\ 2m)\ x^{2c-m-1}\ (1-x^2)^{b-c-m-1/2}\ {}_2F_1(-m-\tfrac{1}{2},\ b-m;\ c-m;\ x^2),$$

(46) $$D^{2m}[x\ (1-x^2)^{b+m-(3/2)}\ {}_2F_1(c+m-3\tfrac{1}{2},\ b;\ c;\ x^2)] =$$

$$=(-1)^m 2^{2m}\ (-\tfrac{1}{2},m)(3\tfrac{1}{2},m)\frac{(c-b,m)}{(c,m)}\ x\ (1-x^2)^{b-m-(3/2)}{}_2F_1(c-3\tfrac{1}{2}, b; c+m; x^2),$$

(47) $$D^{2m}[x^{2c-2}\ (1-x^2)^{b-c+m+1/2}\ {}_2F_1(m+\tfrac{1}{2},\ b;\ c;\ x^2)] =$$

$$= (2-2c,\ 2m)\ x^{2c-2m-2}\ (1-x^2)^{b-c-m+1/2}\ {}_2F_1(-m+\tfrac{1}{2},\ b-m;\ c-m;\ x^2),$$

(48) $$D^{2m}\left[(1-x^2)^{b+m-\frac{1}{2}}\,{}_2F_1(c+m-\tfrac{1}{2},\ b;\ c;\ x^2)\right]=$$

$$=(-1)^m\, 2^{2m}\, (\tfrac{1}{2},\ m)^2\, \frac{(c-b,\ m)}{(c,\ m)}\, (1-x^2)^{b-m-\frac{1}{2}}\, {}_2F_1(c-\tfrac{1}{2},\ b;\ c+m;\ x^2),$$

(49) $$D^{2m+1}\left[x^{2c-1}(1-x^2)^{b-c+m+\frac{1}{2}}\,{}_2F_1(m+\tfrac{1}{2},\ b;\ c;\ x^2)\right]=$$

$$=-(1-2c,\, 2m+1)\, x^{2c-2m-2}\, (1-x^2)^{b-c-m-\frac{1}{2}}\, {}_2F_1(-m-\tfrac{1}{2},\, b-m;\, c-m;\, x^2),$$

(50) $$D^{2m+1}\left[x\,(1-x^2)^{b+m-\frac{1}{2}}\,{}_2F_1(c+m-\tfrac{1}{2},\ b;\ c;\ x^2)\right]=$$

$$=(-1)^m\, 2^{2m}\, (\tfrac{1}{2}, m)(3\tfrac{1}{2}, m)\, \frac{(c-b, m)}{(c, m)}\, (1-x^2)^{b-m-(3/2)}\, {}_2F_1(c-3\tfrac{1}{2}, b; c+m; x^2),$$

(51) $$D^{2m+1}\left[x^{2c-2}\,(1-x^2)^{b-c+m+\frac{1}{2}}\,{}_2F_1(m+\tfrac{1}{2},\ b;\ c;\ x^2)\right]=$$

$$=-\,(2-2c,\ 2m+1)\, x^{2c-2m-3}\, (1-x^2)^{b-c-m-\frac{1}{2}}\,\cdot$$

$$\cdot\, {}_2F_1(-m-\tfrac{1}{2},\ b-m-1;\ c-m-1;\ x^2),$$

(52) $$D^{2m+1}\left[(1-x^2)^{b+m-\frac{1}{2}}\,{}_2F_1(c+m-\tfrac{1}{2},\ b;\ c;\ x^2)\right]=$$

$$=(-1)^m\, 2^{2m}\, (\tfrac{1}{2},\ m)\, (3\tfrac{1}{2},\ m)\, \frac{(c-b,\ m+1)}{(c,\ m+1)}\, x\, (1-x^2)^{b-m-(3/2)}\,\cdot$$

$$\cdot\, {}_2F_1(c-\tfrac{1}{2},\ b;\ c+m+1;\ x^2),$$

(53) $$D^{2m}\left[x(1-x^2)^{a+b-(3/2)}\,{}_2F_1(a,\ b;\ 3/2;\ x^2)\right]=$$

$$=2^{2m}\, (-a+3\tfrac{1}{2},\ m)(-b+3\tfrac{1}{2},\ m)\, x\, (1-x^2)^{a+b-2m-(3/2)}\,\cdot$$

$$\cdot\, {}_2F_1(a-m,\ b-m;\ 3\tfrac{1}{2};\ x^2),$$

(54) $$D^{2m}\left[(1-x^2)^{a+b-\frac{1}{2}}\,{}_2F_1(a,\ b;\ \tfrac{1}{2};\ x^2)\right]=$$

$$=2^{2m}\, (-a+\tfrac{1}{2},\ m)(-b+\tfrac{1}{2},\ m)(1-x^2)^{a+b-2m-\frac{1}{2}}\,\cdot$$

$$\cdot\, {}_2F_1(a-m,\ b-m;\ \tfrac{1}{2};\ x^2),$$

(55) $$D^{2m+1}\left[(1-x^2)^{a+b-\frac{1}{2}}\,{}_2F_1(a,\ b;\ \tfrac{1}{2};\ x^2)\right]=$$

$$=2^{2m+2}\, (-a+\tfrac{1}{2},\ m+1)(-b+\tfrac{1}{2},\ m+1)\, x\, (1-x^2)^{a+b-2m-(3/2)}\,\cdot$$

$$\cdot\, {}_2F_1(a-m,\ b-m;\ 3\tfrac{1}{2};\ x^2),$$

(56) $$D^{2m+1}[x(1-x^2)^{a+b-(3/2)}\,{}_2F_1(a,\ b;\ 3\tfrac{1}{2};\ x^2)] =$$

$$= 2^{2m}(-a+3\tfrac{1}{2},\ m)(-b+3\tfrac{1}{2},\ m)(1-x^2)^{a+b-2m-(5/2)}\ \cdot$$

$$\cdot\ {}_2F_1(a-m-1,\ b-m-1;\ \tfrac{1}{2};\ x^2),$$

(57) $$D^{2m}[(1-x^2)^m\,{}_2F_1(1,\ b;\ \tfrac{1}{2};\ x^2)] =$$

$$= (-1)^m\,2^{2m}\,m!\,(-b+\tfrac{1}{2},\ m)\,{}_2F_1(m+1,\ b;\ \tfrac{1}{2};\ x^2),$$

(58) $$D^{2m}[(1-x^2)^{a+m-1}\,{}_2F_1(a,\ a-\tfrac{1}{2};\ \tfrac{1}{2};\ x^2)] =$$

$$= (-1)^m\,2^{2m}(a,\ m)(1-a,\ m)(1-x^2)^{a-m-1}\,{}_2F_1(a,\ a-\tfrac{1}{2};\ \tfrac{1}{2};\ x^2),$$

(59) $$D^{2m+1}[(1-x^2)^{m+1}\,{}_2F_1(1,\ b;\ \tfrac{1}{2};\ x^2)] =$$

$$= (-1)^{m+1}\,2^{2m+2}(m+1)!\,(-b+\tfrac{1}{2},\ m+1)\,x\,{}_2F_1(m+2,\ b;\ 3\tfrac{1}{2};\ x^2),$$

(60) $$D^{2m+1}[(1-x^2)^{a+m}\,{}_2F_1(a,\ a+\tfrac{1}{2};\ \tfrac{1}{2};\ x^2)] =$$

$$= (-1)^{m+1}\,2^{m+2}(a,\ m+1)(-a,\ m+1)\,x\,(1-x^2)^{a-m-1}\ \cdot$$

$$\cdot\ {}_2F_1(a+1,\ a+\tfrac{1}{2};\ 3\tfrac{1}{2};\ x^2),$$

(61) $$D^{2m}[x(1-x^2)^{a+m-1}\,{}_2F_1(a,\ a+\tfrac{1}{2};\ 3\tfrac{1}{2};\ x^2)] =$$

$$= (-1)^m\,2^{2m}(a,\ m)(1-a,\ m)\,x\,(1-x^2)^{a-m-1}\,{}_2F_1(a,\ a+\tfrac{1}{2};\ 3\tfrac{1}{2};\ x^2),$$

(62) $$D^{2m+1}[x(1-x^2)^{a+m-1}\,{}_2F_1(a,\ a-\tfrac{1}{2};\ 3\tfrac{1}{2};\ x^2)] =$$

$$= (-1)^m\,2^{2m}(a,\ m)(2-a,\ m)(1-x^2)^{a-m-2}\,{}_2F_1(a-1,\ a-\tfrac{1}{2};\ \tfrac{1}{2};\ x^2).$$

Tutte le precedenti formule si possono specializzare per funzioni di KUMMER. Basta porre $x/\sqrt{b}$ in luogo di x, osservando che $D_{x/\sqrt{b}} = \sqrt{b}\,D_x$, e passare al limite per $b \to \infty$. Si ottengono le seguenti formule, solo in parte stabilite nella Nota [7]:

$$D^{2m}[x^{2c-1}\,{}_1F_1(c-m+\tfrac{1}{2};\ c;\ x^2)] = (1-2c,\ 2m)\,x^{2c-2m-1}\,{}_1F_1(c+\tfrac{1}{2};\ c-m;\ x^2),$$

$$D^{2m}[x\,{}_1F_1(-m+3\tfrac{1}{2};\ c;\ x^2)] =$$

$$= (-1)^m\,2^{2m}\frac{(-\frac{1}{2},\ m)\,(3\frac{1}{2},\ m)}{(c,\ m)}\ x\,{}_1F_1(m+3\tfrac{1}{2};\ c+m;\ x^2),$$

$$D^{2m}[x^{2c-2}\,{}_1F_1(c-m-\tfrac{1}{2};\ c;\ x^2)] =$$

$$= (2-2c,\ 2m)\,x^{2c-2m-2}\,{}_1F_1(c-\tfrac{1}{2};\ c-m;\ x^2),$$

$$D^{2m}\,{}_1F_1(-m+\tfrac{1}{2};\ c;\ x^2) =$$

$$= (-1)^m\,2^{2m}\frac{(\frac{1}{2},\ m)^2}{(c,\ m)}\ {}_1F_1(m+\tfrac{1}{2};\ c+m;\ x^2),$$

$$D^{2m+1}[x^{2c-1}\,{}_1F_1(c-m-\tfrac{1}{2};\ c;\ x^2)] =$$

$$= -(1-2c,\ 2m+1)\,x^{2c-2m-2}\,{}_1F_1(c+\tfrac{1}{2};\ c-m;\ x^2),$$

$$D^{2m+1}[x\,{}_1F_1(-m+\tfrac{1}{2};\ c;\ x^2)] =$$

$$= (-1)^m\,2^{2m}\frac{(\frac{1}{2},\ m)(3\frac{1}{2},\ m)}{(c,\ m)}\ {}_1F_1(m+3\tfrac{1}{2};\ c+m;\ x^2),$$

$$D^{2m+1}[x^{2c-2}\,{}_1F_1(c-m-\tfrac{1}{2};\ c;\ x^2)] =$$

$$= -(2-2c,\ 2m+1)\,x^{2c-2m-3}\,{}_1F_1(c-\tfrac{1}{2};\ c-m-1;\ x^2),$$

$$D^{2m+1}\,{}_1F_1(-m+\tfrac{1}{2};\ c;\ x^2) =$$

$$= (-1)^m\,2^{2m}\frac{(\frac{1}{2},\ m)\,(3\frac{1}{2},\ m)}{(c,\ m+1)}\ x\,{}_1F_1(m+3\tfrac{1}{2};\ c+m+1;\ x^2),$$

$$D^{2m}[x\,{}_1F_1(a;\ 3\tfrac{1}{2};\ x^2)] = 2^{2m}\,(a,\ m)\,x\,{}_1F_1(a+m;\ 3\tfrac{1}{2};\ x^2),$$

$$D^{2m}\,{}_1F_1(a;\ \tfrac{1}{2};\ x^2) = 2^{2m}\,(a,\ m)\,{}_1F_1(a+m;\ \tfrac{1}{2};\ x^2),$$

$$D^{2m+1}[x\,{}_1F_1(a;\ 3\tfrac{1}{2};\ x^2)] = 2^{2m}\,(a,\ m)\,{}_1F_1(a+m;\ \tfrac{1}{2};\ x^2),$$

$$D^{2m+1}\,{}_1F_1(a;\ \tfrac{1}{2};\ x^2) = 2^{2m+2}\,(a,\ m+1)\,x\,{}_1F_1(a+m+1;\ 3\tfrac{1}{2};\ x^2),$$

$$D^{2m}[x^{2c-1}\,e^{-x^2}\,{}_1F_1(m-\tfrac{1}{2};\ c;\ x^2)] =$$

$$= (1-2c,\ 2m)\,x^{2c-2m-1}\,e^{-x^2}\,{}_1F_1(-m-\tfrac{1}{2};\ c-m;\ x^2),$$

$$D^{2m}\left[x\, e^{-x^2}\,{}_1F_1(c+m-3\tfrac{1}{2};\ c;\ x^2)\right]=$$

$$=2^{2m}\frac{(-\tfrac{1}{2},\ m)\,(3\tfrac{1}{2},\ m)}{(c,\ m)}\,x\, e^{-x^2}\,{}_1F_1(c-3\tfrac{1}{2};\ c+m;\ x^2),$$

$$D^{2m}\left[x^{2c-2}\, e^{-x^2}\,{}_1F_1(m+\tfrac{1}{2};\ c;\ x^2)\right]=$$

$$=(2-2c,\ 2m)\,x^{2c-2m-2}\, e^{-x^2}\,{}_1F_1(-m+\tfrac{1}{2};\ c-m;\ x^2),$$

$$D^{2m}\left[e^{-x^2}\,{}_1F_1(c+m-\tfrac{1}{2};\ c;\ x^2)\right]=$$

$$=2^{2m}\frac{(\tfrac{1}{2},\ m)}{(c,\ m)}\,e^{-x^2}\,{}_1F_1(c-\tfrac{1}{2};\ c+m;\ x^2),$$

$$D^{2m+1}\left[x^{2c-1}\, e^{-x^2}\,{}_1F_1(m+\tfrac{1}{2};\ c;\ x^2)\right]=$$

$$=-(1-2c,\ 2m+1)\,x^{2c-2m-2}\, e^{-x^2}\,{}_1F_1(-m-\tfrac{1}{2};\ c-m;\ x^2),$$

$$D^{2m+1}\left[x\, e^{-x^2}\,{}_1F_1(c+m-\tfrac{1}{2};\ c;\ x^2)\right]=$$

$$=2^{2m}\frac{(\tfrac{1}{2},\ m)\,(3\tfrac{1}{2},\ m)}{(c,\ m)}\,e^{-x^2}\,{}_1F_1\big(c-3\tfrac{1}{2};\ c+m;\ x^2\big),$$

$$D^{2m+1}\left[x^{2c-2}\, e^{-x^2}\,{}_1F_1(m+\tfrac{1}{2};\ c;\ x^2)\right]=$$

$$=-(2-2c,\ 2m+1)\,x^{2c-2m-3}\, e^{-x^2}\,{}_1F_1(-m-\tfrac{1}{2};\ c-m-1;\ x^2)\,,$$

$$D^{2m+1}\left[e^{-x^2}\,{}_1F_1(c+m-\tfrac{1}{2};\ c;\ x^2)\right]=$$

$$=-2^{2m}\frac{(\tfrac{1}{2},\ m)\,(3\tfrac{1}{2},\ m)}{(c,\ m+1)}\,x\, e^{-x^2}\,{}_1F_1(c-\tfrac{1}{2};\ c+m+1;\ x^2)\,,$$

$$D^{2m}\left[x\, e^{-x^2}\,{}_1F_1(a;\ 3\tfrac{1}{2};\ x^2)\right]=$$

$$=(-1)^m\,2^{2m}\big(-a+3\tfrac{1}{2},\ m\big)\,x\, e^{-x^2}\,{}_1F_1(a-m;\ 3\tfrac{1}{2};\ x^2),$$

$$D^{2m}\left[e^{-x^2}\,{}_1F_1(a;\ \tfrac{1}{2};\ x^2)\right]=$$

$$=(-1)^m\,2^{2m}\,(-a+\tfrac{1}{2},\ m)\,e^{-x^2}\,{}_1F_1(a-m;\ \tfrac{1}{2};\ x^2),$$

$$D^{2m+1}\left[e^{-x^2}\,{}_1F_1(a;\ \tfrac{1}{2};\ x^2)\right]=$$

$$=(-1)^{m+1}\,2^{2m+2}\,(-a+\tfrac{1}{2},\ m+1)\,x\, e^{-x^2}\,{}_1F_1(a-m;\ 3\tfrac{1}{2};\ x^2),$$

$$\mathrm{D}^{2m+1}\left[x\, e^{-x^2}\, {}_1F_1(a;\ 3\tfrac{1}{2};\ x^2)\right] =$$

$$= (-1)^m\, 2^{2m}\, (-a + 3\tfrac{1}{2},\ m)\, e^{-x^2}\, {}_1F_1(a - m - 1;\ \tfrac{1}{2};\ x^2),$$

$$\mathrm{D}^{2m}\, {}_1F_1(1;\ \tfrac{1}{2};\ x^2) = 2^{2m}\, m!\ {}_1F_1(m+1;\ \tfrac{1}{2};\ x^2),$$

$$\mathrm{D}^{2m+1}\, {}_1F_1(1;\ \tfrac{1}{2};\ x^2) = 2^{2m+2}\, (m+1)!\ x\, {}_1F_1(m+2;\ 3\tfrac{1}{2};\ x^2).$$

6. – Proseguendo nello stesso piano di ricerca si potrebbero calcolare derivate, rispetto alla variabile x, di funzioni di GAUSS ad argomento x^3. E basterebbe applicare le nostre formule su operatori differenziali [5]:

$$\mathrm{D}_x^{3m} = \underleftarrow{3^{3m}\ x^2\ \mathrm{D}_{x^3}^m\ x^{3m-1}\ \mathrm{D}_{x^3}^m\ x^{3m-1}\ \mathrm{D}_{x^3}^m},$$

$$\mathrm{D}_x^{3m+1} = \underleftarrow{3^{3m+1}\ \mathrm{D}_{x^3}^m\ x^{3m+1}\ \mathrm{D}_{x^3}^m\ x^{3m+1}\ \mathrm{D}_{x^3}^{m+1}},$$

$$\mathrm{D}_x^{3m+2} = \underleftarrow{3^{3m+2}\ \mathrm{D}_{x^3}^m\ x^{3m+2}\ \mathrm{D}_{x^3}^{m+1}\ x^{3m+2}\ \mathrm{D}_{x^3}^{m+1}}.$$

La ricerca è stata fatta per funzioni di KUMMER nella Nota [7] alla quale rimandiamo; e lasciamo allo studioso la ricerca più generale per funzioni di GAUSS.

Lavori consultati.

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[6] L. Toscano, *Operatori q-permutabili di secondo ordine*, Annuario Liceo « G. La Farina » di Messina (1963), 219-254.

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Summary.

In this paper some differential formulas on the hypergeometric functions of Gauss are established.

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