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On Absolute Cesàro Summability of Laplace Series. (**)

1. – Let $f(\theta, \varphi)$ be defined for the range $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$ and integrable (L) on the surface of a sphere S. The Laplace series associated with this function is

$$f(\theta, \varphi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n + \tfrac{1}{2}) \iint_S f(\theta', \varphi') P_n(\cos r) \sin\theta' \, d\theta' \, d\varphi' , \tag{1.1}$$

where

$$\cos r = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varphi - \varphi')$$

and $P_n(x)$ is the Legendre polynomial.

A generalised mean value of $f(\theta, \varphi)$ is given by:

$$f(r) = \frac{1}{2\pi \sin r} \int_{C_r} f(\theta', \varphi') \sin\theta' \, d\theta' \, d\varphi', \tag{1.2}$$

where the integral is taken along the small circle C_r whose centre is (θ, φ) on the sphere and whose curvilinear radius is r. The series in (1.1) reduces to

$$\sum_{n=0}^{\infty} (n + \tfrac{1}{2}) \int_0^{\pi} f(r) P_n(\cos r) \sin r \, dr \equiv \sum_{n=0}^{\infty} A_n . \tag{1.3}$$

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We write:

$$F_0(x) = f(x), \qquad F_p(x) = \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1} f(t)\, dt\,,$$

$$f_p(x) = \Gamma(p+1)\; x^{-p}\; F_p(x) \qquad (p \geqslant 0)\,,$$

$$F_p(x) = \frac{d}{dx} F_{p+1}(x) \qquad (-1 < p < 0)\,.$$

The object of the present paper is to prove the following new theorems on the absolute CESÀRO summability of the series (1.1).

Theorem 1. *If $f(r)$ is of bounded variation in (η, π), where*

$$\eta = \frac{\mu}{n^{\Delta}}\,, \qquad \frac{2-\alpha}{1+\alpha} < \Delta < 1\,, \qquad 1 > \alpha > \tfrac{1}{2}\,,$$

μ is a large constant, and if

(1.4) $$F_1(t) = \int_0^t |\, f(r)\,|\, dr = O(t^{1+2\alpha}) \qquad as \quad t \to 0,$$

then the series (1.1) *is summable* $|\, C, \alpha + \tfrac{1}{2}\,|$.

Theorem 2. *If $f(r)$ is of bounded variation in (η, π), where*

$$\eta = \frac{\mu}{n^{\Delta}}\,, \qquad 1 > \Delta > \frac{1}{1+\beta-\alpha}\,, \qquad 1 > \beta > \alpha > \tfrac{1}{2}\,,$$

μ is a large constant, and if

(1.5) $$F_{\alpha}(t) = O(t^{1+\beta}) \qquad as \quad t \to 0\,,$$

then the series (1.1) *is summable* $|\, C, \alpha + \tfrac{1}{2}\,|$

2. – To prove the theorems we require the following lemmas:

Lemma 1. *If $S_n^k(r)$ denotes the n-th Cesàro mean of order $k > 1$ of the* *sequence* $\{(n+\frac{1}{2})\,P_n(\cos r)\}$, *then for* $p \geqslant 0$ *we have*

$$(2.1)\qquad \frac{d^p}{dr^p}\,S_n^k(r) = \begin{cases} O(n^{1+p}) & \text{for } 0 \leqslant r \leqslant \pi \\ O\left(\dfrac{n^{p+\frac{1}{2}-k}}{r^{k+\frac{1}{2}}}\right) & \text{for } 0 < r < \pi, \quad [p]+3/2 \geqslant k. \end{cases}$$

For the proof see [1] for the case $\lambda = \frac{1}{2}$.

Lemma 2. *For* $0 < \theta \leqslant \pi$ *we have uniformly*

$$(2.2)\qquad S_n^k(r) = O(n^{1-k}) + O(1/n^2).$$

For the proof see [1] for the case $\lambda = \frac{1}{2}$.

Lemma 3. *We have* ([2], p. 308)

$$(2.3)\qquad (n+1)\,P_n(x) = \frac{d}{dx}\,P_{n+1}(x) - x\,\frac{d}{dx}\,P_n(x)\,.$$

3. - Proof of the Theorem 1. We have, by Lemma 3,

$$n\,A_n = n(n+\tfrac{1}{2})\int_0^\pi f(r)\,P_n(\cos r)\sin r\,dr =$$

$$= n\int_0^\pi f(r)\,(\sin r)\left\{\left[\frac{d}{dx}\,P_{n+1}(x) - x\,\frac{d}{dx}\,P_n(x)\right]_{x=\cos r} - \tfrac{1}{2}\,P_n(\cos r)\right\}dr$$

(by Lemma 3)

$$= n\int_0^\pi f(r)\left[(\cos r)\,\frac{d}{dr}\,P_n(\cos r) - \frac{d}{dr}\,P_{n+1}(\cos r) - \tfrac{1}{2}\,P_n(\cos r)\sin r\right]dr =$$

$$= \int_0^\pi f(r)\Big[(n+\tfrac{1}{2})\,(\cos r)\,\frac{d}{dr}\,P_n(\cos r) - \tfrac{1}{2}\,(\cos r)\,\frac{d}{dr}\,P_n(\cos r)$$

$$- (n+1+\tfrac{1}{2})\,\frac{d}{dr}\,P_{n+1}(\cos r) + \frac{3}{2}\,\frac{d}{dr}\,P_{n+1}(\cos r) - \frac{1}{2}\,(n+\tfrac{1}{2})\,P_n(\cos r)\sin r$$

$$+ \frac{1}{4}\,P_n(\cos r)\sin r\Big]\,dr\,,$$

thus we have

$$(3.1)\quad \begin{cases} T_n^{\alpha+\frac12}(r) = \int_0^\pi \left[\frac{\mathrm{d}}{\mathrm{d}r} S_n^{\alpha+\frac12}(r)\right] f(r)\cos r\,\mathrm{d}r - \frac12\int_0^\pi \sigma_n^{\alpha+\frac12}(r) f(r)\cos r\,\mathrm{d}r \\ \qquad - \int_0^\pi \left[\frac{\mathrm{d}}{\mathrm{d}r} S_{n+1}^{\alpha+\frac12}(r)\right] f(r)\,\mathrm{d}r + \frac32\int_0^\pi \sigma_{n+1}^{\alpha+\frac12}(r) f(r)\,\mathrm{d}r \\ \qquad - \frac12\int_0^\pi S_n^{\alpha+\frac12}(r) f(r)\sin r\,\mathrm{d}r + \frac14\int_0^\pi l_n^{\alpha+\frac12} f(r)\sin r\,\mathrm{d}r \\ \qquad = A + B + C + D + E + F, \end{cases}$$

say, where

$$T_n^k(r), \qquad S_n^k(r), \qquad \sigma_n^k(r), \qquad l_n^k(r),$$

denote the n-th CESÀRO mean of the order k of the sequences

$$\{n\,A_n\}, \qquad \{(n+\tfrac12)\,P_n(\cos r)\}, \qquad \left\{\frac{\mathrm{d}}{\mathrm{d}r} P_n(\cos r)\right\}, \qquad \{P_n(\cos r)\}$$

respectively.

In order to prove the theorem we have to show that

$$(3.2)\qquad \sum_{n=0}^{\infty} n^{-1}\,|\,T_n^{\alpha+\frac12}\,| < \infty .$$

We shall now denote $\frac{\mathrm{d}^p}{\mathrm{d}r^p} S_n^{\alpha+\frac12}(r)$ by $S_n^{(p)}(r)$. We first consider

$$(3.3)\qquad C = \int_0^\pi S_{n+1}^{(1)}(r) f(r)\,\mathrm{d}r = \left[\int_0^\eta + \int_\eta^\pi\right] S_{n+1}^{(1)}(r) f(r)\,\mathrm{d}r = C_1 + C_2,$$

say. Now

$$C_1 = [S_{n+1}^{(1)}(r)\,F_1(r)]_0^\eta - \int_0^\eta S_{n+1}^{(2)}(r)\,F_1(r)\,\mathrm{d}r = C_{1,1} - C_{1,2},$$

say, where

$$(3.4)\qquad C_{1,1} = O(n^{1-\alpha}\eta^\alpha) = O(n^{\frac12-\alpha}).$$

We now have

$$C_{1.2} = \int_0^\eta S^{(2)}_{n+1}(r)\, F_1(r)\, \mathrm{d}r = \left[\int_0^{1/n} + \int_{1/n}^\eta\right] S^{(2)}_{n+1}(r)\, F_1(r)\, \mathrm{d}r = C_{1.2.1} + C_{1.2.2}\,,$$

say. Now

$$C_{1.2.1} = \int_0^{1/n} O(n^3)\, r^{1+2\alpha}\, \mathrm{d}r = O(n^{1-2\alpha})\,. \tag{3.5}$$

Also

$$C_{1.2.2} = \int_{1/n}^{\eta} O\left(\frac{n^{2-\alpha}}{r^{1+\alpha}}\, r^{1+2\alpha}\right) \mathrm{d}r = n^{2-\alpha}\,(\eta^{1+\alpha} - n^{-1-\alpha}) = O(n^{1-2\alpha})\,. \tag{3.6}$$

Finally we have

$$\begin{cases} C_2 = \int_\eta^\pi S^{(1)}_{n+1}(r)\, f(r)\, \mathrm{d}r = \\ \quad = [f(r)\, S^{\alpha+\frac12}_{n+1}(r)]_\eta^\pi - \int_\eta^\pi S^{\alpha+\frac12}_{n+1}(r)\, \mathrm{d}f(r) = O(n^{\frac12-\alpha})\,, \end{cases} \tag{3.7}$$

by Lemma 2. Combining (3.4), (3.5), (3.6) and (3.7) we see that

$$C = O(n^{\frac12-\alpha})\,. \tag{3.8}$$

Clearly the order estimates for A, B, D, E and F are the same as those for C, and in view of (3.8) we see that (3.2) holds.

Thus the Theorem 1 is completely established.

4. - Proof of Theorem 2. Proceeding as in the proof of the Theorem 1, it is sufficient to show that

$$\sum n^{-1}\,|\,C\,| < \infty$$

under the condition of the theorem. We have

$$C = \int_0^\eta S^{(1)}_{n+1}(r)\, f(r)\, \mathrm{d}r = [S^{(1)}_{n+1}(r)\, F_1(r)]_0^\eta - \int_0^\eta S^{(2)}_{n+1}(r)\, F_1(r)\, \mathrm{d}r = J_1 - J_2\,,$$

say. It is

$$(4.1) \qquad J_1 = [S^{(1)}_{n+1}(r)\, F_1(r)]_0^\eta = [n^{1-\alpha}\ \eta^{-1-\alpha}\ \eta^{2+\beta-\alpha}] = O(n^{\alpha-\beta})\,.$$

Also

$$J_2 = \int_0^\eta S^{(2)}_{n+1}(r)\, F_1(r)\, dr = \int_0^\eta S^{(2)}_{n+1}(r)\, dr\, \frac{1}{\Gamma(1-\alpha)} \int_0^r (r-u)^{-\alpha} F_\alpha(u)\, du$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^\eta F_\alpha(u) \left[\int_u^\eta (r-u)^{-\alpha} S^{(2)}_{n+1}(r)\, dr \right] du$$

$$= \int_0^\eta F_\alpha(u)\ F(\eta,\ u)\, du = \left[\int_0^{1/n} + \int_{1/n}^\eta \right] F_\alpha(u)\ F(\eta,\ u)\, du = J_{2,1} + J_{2,2}\,,$$

say. Now

$$F(\eta,\ u) = \int_u^\eta (r-u)^{-\alpha} S^{(2)}_{n+1}(r)\, dr = \left[\int_u^{2u} + \int_{2u}^\eta \right] (r-u)^{-\alpha} S^{(2)}_{n+1}(r)\, dr$$

$$= \int_u^{2u} (r-u)^{-\alpha} O(n^3)\, dr + u^{-\alpha} \int_{2u}^\xi S^{(2)}_{n+1}(r)\, dr \qquad (2u \leqslant \xi \leqslant \eta)$$

$$= O(n^3\ u^{1-\alpha}) + O(n^2\ u^{-\alpha})\,.$$

Again, if $u + (1/n) < \eta$,

$$F(\eta,\ u) = \left[\int_u^{u+(1/n)} + \int_{u+(1/n)}^\eta \right] (r-u)^{-\alpha} \cdot S^{(2)}_{n+1}(r)\, dr = F' + F''\,,$$

$$F' = \int_u^{u+(1/n)} (r-u)^{-\alpha}\, O\!\left(\frac{n^{2-\alpha}}{r^{1+\alpha}}\right) dr = O(n\ u^{-1-\alpha})\,,$$

$$F'' = \int_{u+(1/n)}^\eta (r-u)^{-\alpha} S^{(2)}_{n+1}(r)\, dr = O(n\ u^{-1-\alpha})\,,$$

thus

$$F(\eta,\ u) = O(n\ u^{-1-\alpha})\,.$$

Now we have

$$(4.2) \qquad J_{2,1} = \int_0^{1/n} O(n^3 u^{1-\alpha})\ u^{1+\beta}\, du + \int_0^{1/n} O(n^2 u^{-\alpha})\ u^{1+\beta}\, du = O(n^{\alpha-\beta})\,.$$

Also

$$(4.3) \qquad J_{2,2} = \int_{1/n}^{\eta} O(n\, u^{-1-\alpha})\ u^{1+\beta}\, du = [\eta^{1-\Delta(1+\beta-\alpha)} - n^{\alpha-\beta}] = O(n^{\alpha-\beta})\,,$$

therefore we get

$$(4.4) \qquad C = O(n^{\alpha-\beta})\,,$$

thus

$$(4.5) \qquad \sum n^{-1} \,|\, C \,| < \infty\,.$$

The Theorem 2 is proved.

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References.

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[2] E. T. WHITTAKER and G. N. WATSON, *A Course of Modern Analysis*, University Press, Cambridge 1950.

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