Riv. Mat. Univ. Parma (2) 8 (1967), 219-220

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On Quasi-Frobenius Local Rings. (**)

A ring R is quasi-FROBENIUS ($=QF$) in case every right ideal is an annihilator right ideal, every left ideal is an annihilator left ideal and R is right (or left) Artinian. QF rings have many interesting characterizations. Recently FAITH [2] characterized QF rings as rings R such that every projective right R-module is injective. FAITH and WALKER [3] proved that a ring R is QF if every injective right R-module is projective. It is stated in the review [MR 33 # 1329] of a paper [5] of TOL'SKAJA that a ring R is quasi-FROBENIUS if every injective right R-module is free. We note that a simple counter example exists to this result. Further we prove that a ring R is quasi-FROBENIUS local if every injective right R-module is free.

Counter example. Let R be a semi-simple Artinian ring such that R is not a division ring. R is quasi-FROBENIUS, because every right R-module is injective. Let I be a minimal right ideal of R. $I = eR$ e idempotent, $e \neq 1$. I is an injective right R-module. But I is not a free R-module, because if I be free R-module, then I_R contains a copy of R_R, which is in contradiction to the minimality of I.

Theorem. *A ring R is a quasi-Frobenius local ring R if every injective right R-module is free.*

Proof. Every injective R-module over a quasi FROBENIUS ring R is projective [3]. Every projective R-module over a local ring R is free, KAPLANSKY ([4], theorem 2, page 374). Hence every injective R-module over a quasi-FROBENIUS local ring R is free.

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(**) Ricevuto: 16-IX-1967.

Conversely let R be such that every injective right R-module is free. As a free module is projective, therefore R is QF [3]. Hence R is Artinian. Now to prove that R is local it is sufficient to show that R has no idempotents except 1 ([1], page 371). Let if possible $e \neq 1$ be an idempotent in R. Let I be a minimal right ideal in the collection of right ideals generated by an idempotent. As $I = e_1 R$ for idempotent e_1, I is a projective R-module and therefore is injective R-module [2]. Hence I is free. Consequently I_R contains a direct summand isomorphic to R_R. But R_R has a proper direct summand. Therefore I_R has proper direct summand. A proper direct summand of I_R is a right ideal of R generated by an idempotent and is properly contained in I, which is a contradiction. Hence R is local.

The authors thank Dr. S. K. Jain for his guidance during the preparation of this Note.

References.

[1] C. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience Publishers 1966.

[2] C. Faith, *Rings with ascending condition on annihilators*, Nagoya Math. J. **27** (1966), 179-191.

[3] C. Faith and E. A. Walker, *Direct-sum representations of injective modules*, J. Algebra **5** (1967), 203-221.

[4] I. Kaplansky, *Projective modules*, Ann. of Math. **68** (1958), 372-377.

[5] T. S. Tol'skaja, *Injectivity and freness* (Russian), Sirbirsk Mat. Ž. **6** (1965), 1202-1207.

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