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A Theorem in Operational Calculus and Infinite Integrals. (**)

1. - Introduction.

A function $\Phi(p)$ is operationally related to $h(t)$ when they satisfy the integral equation

$$\Phi(p) = p \int_0^\infty e^{-pt}\, h(t)\, dt\,, \tag{1.1}$$

provided that the integral is convergent and $\operatorname{Re} p > 0$; as usual, we shall denote (1.1) by the symbol

$$\Phi(p) \doteqdot h(t)\,. \tag{1.2}$$

The object of this paper is to prove a theorem in operational calculus and use it to evaluate some infinite double integrals involving LEGENDRE and MEIJER's G-function. Most of the results are believed to be new.

We require the following formulae in the investigation:

$$\left\{\begin{aligned} &\sum_{\varrho,-\varrho}\ \sum_{\mu,-\mu} \frac{\Gamma(-\varrho)\,\Gamma(-\mu)\,\Gamma(\lambda+\mu+\varrho+\delta)}{p^{\mu+\delta+\lambda+\varrho-1}\, 2^{\mu+\varrho+\delta+2}\, \Gamma(\delta+1)}\, \alpha^{\varrho}\, \beta^{\mu}\, \gamma^{\delta}\,\cdot \\ &\cdot F_c\!\left[\tfrac{1}{2}(\lambda+\mu+\varrho+\delta), \tfrac{1}{2}(\lambda+\mu+\varrho+\delta+1);\ \varrho+1, \mu+1,\ \delta+1;\ \frac{\alpha^2}{p^2}, \frac{\beta^2}{p^2}, \frac{\gamma^2}{p^2}\right] \\ &\qquad\qquad \doteqdot x^{\lambda-1} K_{\varrho}(\alpha x)\, K_{\mu}(\beta x)\, I_{\delta}(\gamma x), \end{aligned}\right. \tag{1.3}$$

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valid for $\operatorname{Re}(\lambda + \delta \pm \mu \pm \varrho) > 0$, $\operatorname{Re}(p + \alpha + \beta) > |\operatorname{Re}\gamma|$.

$$(1.4)\quad \begin{cases} \displaystyle\sum_{\varrho,-\varrho}\sum_{\mu,-\mu}\sum_{\delta,-\delta} \frac{\Gamma(-\varrho)\,\Gamma(-\mu)\,\Gamma(-\delta)\,\Gamma(\lambda+\mu+\varrho+\delta)}{2^{\varrho+\mu+\delta+3}\,p^{\lambda+\mu+\varrho+\delta-1}}\,\alpha^{\varrho}\,\beta^{\mu}\,\gamma^{\delta}\cdot \\ \cdot F_c\left[\tfrac{1}{2}(\lambda+\mu+\varrho+\delta),\,\tfrac{1}{2}(\lambda+\mu+\varrho+\delta+1);\,\varrho+1,\,\mu+1,\,\delta+1;\,\frac{\alpha^2}{p^2},\frac{\beta^2}{p^2},\frac{\gamma^2}{p^2}\right] \\ \qquad \doteqdot x^{\lambda-1}K_{\varrho}(\alpha x)\,K_{\mu}(\beta x)\,K_{\delta}(\gamma x)\,, \end{cases}$$

valid for $\operatorname{Re}(\lambda \pm \delta \pm \varrho \pm \mu) > 0$, $\operatorname{Re}(p + \alpha + \beta + \gamma) > 0$. (1.3) and (1.4) has been derived from the formula due to SHARMA ([2] p. 86).

$$(1.5)\quad \begin{cases} \displaystyle\sum_{\varrho,-\varrho} \frac{\Gamma(-\varrho)\,\Gamma(\lambda+\mu+\varrho+\delta)\,\alpha^{\varrho}\,\beta^{\mu}\,\gamma^{\delta}}{\Gamma(\mu+1)\;\Gamma(\delta+1)\,p^{\mu+\delta+\lambda+\varrho-1}\,2^{\varrho+\mu+\delta+1}}\cdot \\ \cdot F_c\left[\tfrac{1}{2}(\lambda+\mu+\varrho+\delta),\,\tfrac{1}{2}(\lambda+\mu+\varrho+\delta+1);\,\varrho+1,\,\mu+1,\,\delta+1;\,\frac{\alpha^2}{p^2},\,\frac{\beta^2}{p^2},\,\frac{\gamma^2}{p^2}\right] \\ \qquad \doteqdot x^{\lambda-1}K_{\varrho}(\alpha x)\,I_{\mu}(\beta x)\,I_{\delta}(\gamma x)\,, \end{cases}$$

valid for $\operatorname{Re}(\lambda + \mu + \delta \pm \varrho) > 0$, $\operatorname{Re}(p+\alpha) > |\operatorname{Re}\beta| + |\operatorname{Re}\gamma|$, on applying the well-known formula

$$(1.6)\qquad K_v(x) = \tfrac{1}{2}\sum_{v,-v}\Gamma(-v)\,\Gamma(v+1)\,I_v(x).$$

Taking $\gamma \to 0$ in (1.3), we obtain the following formula

$$(1.7)\quad \begin{cases} \displaystyle\sum_{\varrho,-\varrho}\sum_{\mu,-\mu} \frac{\Gamma(-\varrho)\Gamma(-\mu)\Gamma(\lambda+\mu+\varrho)}{2^{\mu+\varrho+2}\,p^{\mu+\varrho+\lambda-1}}\,\alpha^{\varrho}\,\beta^{\mu}\cdot \\ \cdot F_4\left[\tfrac{1}{2}(\lambda+\mu+\varrho),\,\tfrac{1}{2}(\lambda+\mu+\varrho+1);\,\varrho+1,\,\mu+1;\,\frac{\alpha^2}{p^2},\,\frac{\beta^2}{p^2}\right] \\ \qquad \doteqdot x^{\lambda-1}\,K_{\varrho}(\alpha x)\,K_{\mu}(\beta x), \end{cases}$$

valid for $\operatorname{Re}(\lambda \pm \mu \pm \varrho) > 0$, $\operatorname{Re}(p + \alpha + \beta) > 0$.

2. – Theorem (a). *If*

$$\Phi(p) \doteqdot h(t)\,, \qquad \Psi(p) \doteqdot K_{\mu}(ct)\,K_{\varrho}(bt)\,h(t)\,,$$

then

$$(2.1)\quad \int_0^\infty\int_0^\infty \cosh\mu\theta \cosh\varrho\varphi\cdot(a+b\cosh\varphi+c\cosh\theta)^{-1}\cdot \Phi(a+b\cosh\varphi+c\cosh\theta)\,d\theta\,d\varphi=\frac{1}{a}\,\Psi(a),$$

provided the integrals are absolutely convergent and Re $b>0$ *and* Re $c>0$.

Proof. By definition, we have

$$\Phi(p)=p\int_0^\infty e^{-px}\,h(x)\,dx,$$

then

$$\int_0^\infty\int_0^\infty \cosh\mu\theta \cosh\varrho\varphi\cdot(a+b\cosh\varphi+c\cosh\theta)^{-1}\,\Phi(a+b\cosh\varphi+c\cosh\theta)\,d\theta\,d\varphi=$$

$$=\int_0^\infty\int_0^\infty \cosh\mu\theta\cosh\varrho\varphi\cdot\left[\int_0^\infty \exp\{-(a+b\cosh\varphi+c\cosh\theta)x\}\,h(x)\,dx\right]d\theta\,d\varphi=$$

$$=\int_0^\infty e^{-ax}\,h(x)\left[\int_0^\infty \cosh\mu\theta\cdot e^{-cx\cosh\theta}\,d\theta\int_0^\infty \cosh\varrho\varphi\cdot e^{-bx\cosh\varphi}\,d\varphi\right]dx=$$

$$=\int_0^\infty e^{-ax}\,K_\mu(cx)\,K_\varrho(bx)\,h(x)\,dx=\frac{1}{a}\;\Psi(a),$$

on changing the order of integration and evaluating the inner integral by means of the formula (Erdélyi [1], p. 183)

$$(2.2)\qquad \int_0^\infty \cosh v\theta\cdot\exp(-a\cosh\theta)\,d\theta=K_v(a),$$

where Re $a>0$. The change of the order of integration can be justified by the application of de la Vallée-Poussin's theorem (cf. [3]) when the integrals involved are absolutely convergent.

3. – Theorem (b). *If*

$$\Phi(p)\doteqdot h(t),\qquad \Psi(p)\doteqdot t^{-1}\,h(t),$$

then

$$(3.1)\quad \left\{\begin{aligned}&\int_0^\infty\int_0^\infty \cosh\tfrac{1}{2}\theta\cosh\tfrac{1}{2}\varphi\cdot(a+b\cosh\varphi+c\cosh\theta)^{-1}\cdot\\ &\cdot\;\Phi(a+b\cosh\varphi+c\cosh\theta)\,d\theta\,d\varphi=\frac{\pi}{2\sqrt{bc}\,(a+b+c)}\;\Psi(a+b+c),\end{aligned}\right.$$

provided the integrals are absolutely convergent and $\operatorname{Re} b > 0$ *and* $\operatorname{Re} c > 0$.

Proof. (3.1) can be proved in the similar manner as (2.1) by using the formula

$$\int_0^\infty \cosh\tfrac{1}{2}\theta \cdot \exp(-a\cosh\theta)\, d\theta = \left(\frac{\pi}{2a}\right)^{1/2} e^{-a}, \tag{3.2}$$

where $\operatorname{Re} a > 0$, instead of (2.2). (3.2) has been deduced from (2.2) by using the well-known formula

$$K_{\pm\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}. \tag{3.3}$$

4. – We now proceed to evaluate a few infinite double integrals by applying the theorems (2.1) and (3.1).

Example I. If we take in Theorem (a) $h(t) = t^{\lambda-1}$, then (ERDÈLYI [1], p. 137, eqn. (1))

$$\Phi(p) = p^{1-\lambda}\,\Gamma(\lambda)\,, \qquad \operatorname{Re}\lambda > 0\,, \qquad \operatorname{Re} p > 0\,, \tag{4.1}$$

and [cf. (1.7)]

$$\left\{\begin{aligned} &\Psi(p) = \sum_{\varrho,-\varrho}\sum_{\mu,-\mu} \frac{\Gamma(-\varrho)\,\Gamma(-\mu)\,\Gamma(\lambda+\mu+\varrho)}{2^{\mu+\varrho+2}\,p^{\lambda+\mu+\varrho-1}}\, b^\varrho c^\mu \cdot \\ &\cdot F_4\left[\tfrac{1}{2}(\lambda+\mu+\varrho),\ \tfrac{1}{2}(\lambda+\mu+\varrho+1);\ \varrho+1,\ \mu+1;\ \frac{b^2}{p^2},\ \frac{c^2}{p^2}\right], \\ &\qquad \operatorname{Re}(\lambda\pm\mu\pm\varrho) > 0\,, \qquad \operatorname{Re}(p+b+c) > 0. \end{aligned}\right. \tag{4.2}$$

Using (4.1) and (4.2) in (2.1), we have

$$\left\{\begin{aligned} &\int_0^\infty\!\!\int_0^\infty \cosh\mu\theta\,\cosh\varrho\varphi\cdot(a+b\cosh\varphi+c\cosh\theta)^{-\lambda}\, d\theta\, d\varphi = \\ &= \sum_{\varrho,-\varrho}\sum_{\mu,-\mu} \frac{\Gamma(-\varrho)\,\Gamma(-\mu)\,\Gamma(\lambda+\mu+\varrho)\, b^\varrho c^\mu}{2^{\mu+\varrho+2}\,a^{\lambda+\mu+\varrho}\,\Gamma(\lambda)} \cdot \\ &\cdot F_4\left[\tfrac{1}{2}(\lambda+\mu+\varrho),\ \tfrac{1}{2}(\lambda+\mu+\varrho+1);\ \varrho+1,\ \mu+1;\ \frac{b^2}{a^2},\ \frac{c^2}{a^2}\right], \end{aligned}\right. \tag{4.3}$$

valid for $\operatorname{Re}(\lambda\pm\mu\pm\varrho) > 0$, $\operatorname{Re} b > \operatorname{Re} a$, $\operatorname{Re} c > \operatorname{Re} a$, $\operatorname{Re} a > 0$. In particular if we take $\lambda = 1/2$ in (4.3), we obtain the following interesting integral:

$$(4.4)\quad \left\{\begin{aligned} &\int_0^\infty\int_0^\infty \cosh\mu\theta \cosh\varrho\varphi\cdot(a + b\cosh\varphi + c\cosh\theta)^{-1/2}\, d\theta\, d\varphi = \\ &= \frac{d^{1/2}\cos\varrho\pi\cos\mu\pi\,\Gamma(\frac{1}{2}-\varrho-\mu)}{\sqrt{\pi}\cos(\mu+\varrho)\pi\,\cos(\mu-\varrho)\pi}\, Q^{-\mu}_{\varrho+1/2}(\cosh\alpha)\; Q^{-\varrho}_{\mu-1/2}(\cosh\beta)\,, \end{aligned}\right.$$

where $\sinh\alpha = c\,d$, $\sinh\beta = b\,d$, $\cosh\alpha\cosh\beta = a\,d$, $|\operatorname{Im}\alpha| < \frac{\pi}{2}$, $|\operatorname{Im}\beta| < \frac{\pi}{2}$, $\operatorname{Re}(\frac{1}{2} \pm \varrho \pm \mu) > 0$, $\operatorname{Re} b > 0$, $\operatorname{Re} c > 0$.

Example 2. If we take in Theorem (a) $h(t) = t^{\alpha-1/2} I_\beta(t)$, then (MacRobert [4], p. 342)

$$(4.5)\quad \Phi(p) = \sqrt{2/\pi}\, p(p^2-1)^{-1/2\alpha}\, Q^\alpha_{\beta-1/2}(p)\ ^{(1)},\quad \operatorname{Re}(\alpha+\beta) > -\tfrac{1}{2},\quad \operatorname{Re} p > 1,$$

$$(4.6)\quad \left\{\begin{aligned} &\Psi(p) = \sum_{\varrho,-\varrho}\sum_{\mu,-\mu} \frac{\Gamma(-\varrho)\,\Gamma(-\mu)\,\Gamma(\alpha+\beta+\varrho+\mu+\frac{1}{2})}{p^{\alpha+\beta+\varrho+\mu-1/2}\;2^{\varrho+\mu+\beta+2}\,\Gamma(\beta+1)}\, b^\varrho c^\mu\cdot \\ &\cdot F_c\left[\tfrac{1}{2}(\alpha+\beta+\varrho+\mu+\tfrac{1}{2}),\, \tfrac{1}{2}(\alpha+\beta+\varrho+\mu+\tfrac{3}{2});\, \varrho+1,\, \mu+1,\, \beta+1;\, \frac{b^2}{p^2}, \frac{c^2}{p^2}, \frac{1}{p^2}\right], \\ &\qquad \operatorname{Re}(\alpha+\beta\pm\varrho\pm\mu+\tfrac{1}{2}) > 0, \qquad \operatorname{Re}(p+b+c) > 1\,. \end{aligned}\right.$$

Using (4.5) and (4.6) in (2.1), we get

$$(4.7)\quad \left\{\begin{aligned} &\int_0^\infty\int_0^\infty \cosh\mu\theta\cosh\varrho\varphi\cdot[(a+b\cosh\varphi+c\cosh\theta)^2-1]^{-1/2\alpha}\,\cdot \\ &\qquad\cdot\, Q^\alpha_{\beta-1/2}(a+b\cosh\varphi+c\cosh\theta)\, d\theta\, d\varphi = \\ &= \sum_{\varrho,-\varrho}\sum_{\mu,-\mu} \frac{\Gamma(-\varrho)\Gamma(-\mu)\Gamma(\alpha+\beta+\varrho+\mu+\frac{1}{2})\sqrt{\pi}}{a^{\alpha+\beta+\varrho+\mu+1/2}\;2^{\varrho+\mu+\beta+5/2}\,\Gamma(\beta+1)}\, b^\varrho c^\mu\cdot \\ &\cdot F_c\left[\tfrac{1}{2}(\alpha+\beta+\varrho+\mu+\tfrac{1}{2}),\, \tfrac{1}{2}(\alpha+\beta+\varrho+\mu+\tfrac{3}{2});\, \varrho+1,\, \mu+1,\, \beta+1;\, \frac{b^2}{a^2}, \frac{c^2}{a^2}, \frac{1}{a^2}\right], \end{aligned}\right.$$

(1) We have used MacRobert's definition of $Q^m_n(x)$ and [cf. (1.3)].

valid for $\mathrm{Re}\,(\alpha+\beta+\varrho+\mu+\frac{1}{2})>0$, $\mathrm{Re}\,a>\mathrm{Re}\,b$, $\mathrm{Re}\,a>\mathrm{Re}\,c$, $\mathrm{Re}\,a>1$.

Example 3. Taking, in Theorem (a), $h(t)=t^{\alpha-\frac{1}{2}}K_\beta(t)$, then (cf. ERDÉLYI [1], p. 198)

$$(4.8)\quad \begin{cases} \Phi(p)=\sqrt{\dfrac{\pi}{2}}\,\Gamma(\alpha+\beta+\frac{1}{2})\,\Gamma(\alpha-\beta+\frac{1}{2})\,p(p^2-1)^{-(\frac{1}{2})\alpha}\,P^{-\alpha}_{\beta-(1/2)}(p)\\[2ex] \qquad\qquad \mathrm{Re}\,(\alpha\pm\beta+\frac{1}{2})>0, \qquad \mathrm{Re}\,(p+1)>0,\end{cases}$$

and [cf. (1.4)]

$$(4.9)\quad \begin{cases} \Psi(p)=\displaystyle\sum_{\mu,-\mu}\sum_{\varrho,-\varrho}\sum_{\beta,-\beta}\frac{\Gamma(-\mu)\Gamma(-\varrho)\,\Gamma(-\beta)\,\Gamma(\alpha+\beta+\mu+\varrho+\frac{1}{2})\,b^\varrho\,c^\mu}{2^{\varrho+\mu+\beta+3}\quad p^{\alpha+\beta+\mu+\varrho-(\frac{1}{2})}}\cdot\\[2ex] \cdot F_c\left[\tfrac{1}{2}(\alpha+\beta+\mu+\varrho+\tfrac{1}{2}),\,\tfrac{1}{2}(\alpha+\beta+\mu+\varrho+\tfrac{3}{2});\,\varrho+1,\,\mu+1,\,\beta+1;\,\dfrac{b^2}{p^2},\dfrac{c^2}{p^2},\dfrac{1}{p^2}\right],\\[2ex] \qquad \mathrm{Re}\,(\alpha\pm\beta\pm\mu\pm\varrho+\frac{1}{2})>0, \qquad \mathrm{Re}\,(p+b+c+1)>0.\end{cases}$$

Using (4.8) and (4.9) in (2.1), we have

$$(4.10)\quad \begin{cases} \displaystyle\int_0^\infty\!\!\int_0^\infty \cosh\mu\theta\,\cosh\varrho\varphi\cdot[(a+b\cosh\varphi+c\cosh\theta)^2-1]^{-(\frac{1}{2})\alpha}\cdot\\ \qquad\qquad\qquad\cdot\,P^{-\alpha}_{\beta-\frac{1}{2}}(a+b\cosh\varphi+c\cosh\theta)\,d\theta\,d\varphi=\\[2ex] =\displaystyle\sum_{\mu,-\mu}\sum_{\varrho,-\varrho}\sum_{\beta,-\beta}\frac{\Gamma(-\mu)\,\Gamma(-\varrho)\,\Gamma(-\beta)\,\Gamma(\alpha+\beta+\mu+\varrho+\frac{1}{2})\,b^\varrho\,c^\mu}{\sqrt{\pi}\;2^{\varrho+\mu+\beta+5/2}\;a^{\alpha+\beta+\mu+\varrho+\frac{1}{2}}\;\Gamma(\alpha+\beta+\frac{1}{2})\;\Gamma(\alpha-\beta+\frac{1}{2})}\cdot\\[2ex] \cdot F_c\left[\tfrac{1}{2}(\alpha+\beta+\mu+\varrho+\tfrac{1}{2}),\,\tfrac{1}{2}(\alpha+\beta+\mu+\varrho+\tfrac{3}{2});\,\varrho+1,\,\mu+1,\,\beta+1;\,\dfrac{b^2}{a^2},\dfrac{c^2}{a^2},\dfrac{1}{a^2}\right],\end{cases}$$

valid for $\mathrm{Re}\,(\alpha\pm\beta\pm\mu\pm\varrho+\frac{1}{2})>0$, $\mathrm{Re}\,a>\mathrm{Re}\,b$, $\mathrm{Re}\,a>\mathrm{Re}\,c$, $\mathrm{Re}\,a>1$.

Example 4. Now we take in Theorem (b) $h(t)=t^{-\lambda}\,G^{\alpha,\beta}_{\gamma,\delta+1}\left[t\,\middle|\,\begin{matrix}a_1,\dots,a_\gamma\\ b_1,\dots,b_\delta,\lambda\end{matrix}\right]$, then (cf. ERDÈLYI [1], p. 222)

$$(4.11)\quad \begin{cases} \Phi(p)=p^\lambda\,G^{\alpha,\beta}_{\gamma,\delta}\left[\dfrac{1}{p}\,\middle|\,\begin{matrix}a_1,\dots,a_\gamma\\ b_1,\dots,b_\delta\end{matrix}\right]\\[2ex] \mathrm{Re}\,\lambda>1+\mathrm{Re}\,b_j \qquad (j=1,2,\dots,\alpha)\\[2ex] \gamma+\delta<2(\alpha+\beta), \qquad |\arg p|<(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)\pi.\end{cases}$$

Using the values of $\Phi(p)$ and $\Psi(p)$ (which can be obtained in the same way) in (3.1), we get

$$(4.12)\quad \begin{cases} \displaystyle\int_0^\infty\int_0^\infty \cosh\mu\theta \cosh\varrho\varphi\cdot(a + b\cosh\varphi + c\cosh\theta)^{\lambda-1}\cdot \\ \qquad\qquad \cdot G^{\alpha,\beta}_{\gamma,\delta}\left[\dfrac{1}{a+b\cosh\varphi+c\cosh\theta}\,\middle|\,\begin{matrix}a_1, \ldots, a_\gamma\\ b_1, \ldots, b_\delta\end{matrix}\right] d\theta\, d\varphi = \\ = \dfrac{\pi}{2\sqrt{bc}\,(a+b+c)^{-\lambda}}\, G^{\alpha,\,\beta+1}_{\gamma+1,\,\delta+1}\left[(a+b+c)^{-1}\,\middle|\,\begin{matrix}\lambda+1, & a_1, \ldots, a_\gamma\\ b_1, \ldots, b_\delta, & \lambda\end{matrix}\right], \end{cases}$$

valid by analytic continuation for $\operatorname{Re} b > 0$, $\operatorname{Re} c > 0$, $0 \leqslant \alpha \leqslant \delta$, $0 \leqslant \beta \leqslant \gamma$, $\gamma + \delta < 2(\alpha + \beta)$, $\operatorname{Re} a > 0$, $\operatorname{Re}(\lambda + b_j) < 0$ $(j = 1, 2, \ldots, \alpha)$.

Since the G-function is a generalization of a great many of the special functions occurring in applied mathematics, (4.12) can yield results involving BESSEL and WHITTAKER functions.

References.

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[4] T. M. MACROBERT, *Spherical Harmonics*, Methuen, London 1947.

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