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SURJEET SINGH (*)

On Tertiary Radicals of an Ideal in a Ring. (**)

1. - Introduction.

LESIEUR and CROISOT defined the concept of a tertiary radical of any left ideal and a second tertiary radical of a two sided ideal of a ring [1]. If A is a two sided ideal in a ring R we call the tertiary radical of A, when A is regarded as a left ideal, as the first tertiary radical, and the tertiary radical of A, defined when A is regarded as a two sided ideal, as second tertiary radical. It is proved in [1] that for any ideal of a ring with descending chain condition (d.c.c.) on left ideals these two tertiary radicals are equal. Analogous definitions and results can be stated for right ideals of a ring. For any ring with zero right singular ideal and ascending chain condition (a.c.c.) on right ideals GOLDIE in ([2], p. 278, Theorem 5.7) proved that the first and second tertiary radical of the zero ideal (0) are equal. GOLDIE denotes the first and second radical of (0) by $\operatorname{rad} R$ and $\operatorname{Rad} R$ respectively. We know that any non-zero right ideal of a ring with a.c.c. on right ideals contains a uniform right ideal. Main purpose of this paper is to prove following generalizations of GOLDIE's, and LESIEUR and CROISOT results.

Theorem. *If R be any ring with zero right singular ideal and every non-zero right ideal of R contains a uniform right ideal, then for any two sided ideal A of R which is a member of family of closed right ideals of R, two tertiary radicals of A are equal.*

Theorem. *If R be any ring in which every non-zero right ideal of R contains a minimal right ideal, then* $\operatorname{rad} R = \operatorname{Rad} R$.

(*) Indirizzo: Department of Mathematics, Kirori Mal College, University of Delhi, Delhi-7, India.

(**) Ricevuto: 30-I-1967.

2. - Preliminary definitions and notations.

By an R-module M we shall always mean a right R-module M. For any $x \in M$, $a \in R$, xR^*a denotes $xRa \cup \{xa\}$. For any $a \in R$, $(a)_r$ and $(a)_t$ denote the right ideal and the two sided ideal of R generated by a respectively. For any ring R, with zero right singular ideal, $L^s(R)$ denotes the lattice of all closed right ideals of R, for any right ideal A, $\mathrm{Cl}(A)$ denotes the closure of A. If N and N' be two submodules of a module M, $N \subset' N'$ denotes that N' is an essential extension of N.

Definition 1. For any submodule N of M its tertiary radical $\mathfrak{R}_1(N)$ is defined to be the set of elements $a \in R$ such that for each $x \in M$, $x \notin N$, there exists y in the submodule generated by x with the property that $y \notin N$ and $yR^*a \subseteq N$. $\mathfrak{R}_1(0)$ is denoted by rad M. If A is any right of R ring R, then $\mathfrak{R}_1(A)$ denotes the radical of A which we define, when A is regarded as an R-submodule of R. The tertiary radical of (0), when (0) is regarded as an R-submodule of A, is denoted by rad A.

Definition 2. If A is any two sided ideal of R, the second tertiary radical $\mathfrak{R}_2(A)$ of A is defined to be the set of elements $a \in R$ such that for each $x \in R$, $x \notin A$, there exists $y \in (x)_t$ with the property that $y \notin A$ and $yR^*a \subseteq A$. We denote by $\mathrm{Rad}\, R$ tertiary radical $\mathfrak{R}_2(0)$.

3. – In this section we give characterisations of the tertiary radical $\mathfrak{R}_1(N)$ of any submodule N of an R-module M, and of the second tertiary radical $\mathfrak{R}_2(A)$ of any two sided ideal A of a ring R.

Proposition 3.1. *Let N be any submodule of M. If $a \in R$, then $a \in \mathfrak{R}_1(N)$ if and only if there exists a submodule P of M containing N such that $Pa \subseteq N$ and $\overline{P} = P - N$ is an essential submodule of the difference module $\overline{M} = M - N$.*

Proof. Let $a \in \mathfrak{R}_1(N)$. Let $P = \{x \in R \mid xR^*a \subseteq N\}$. P is a submodule of M containing N. We prove that $\overline{P}$ is essential in $\overline{M}$. Let $\overline{K} = K - N$ be any non-zero submodule of $\overline{M}$. Consider any $x (\neq \overline{0}) \in \overline{K}$, then $x \in K$ and $x \notin N$. Thus by definition of $\mathfrak{R}_1(N)$, there exists y in the submodule generated by x such that $y \notin N$ and $yR^*a \subseteq N$. Then $\overline{y} \in \overline{K} \cap \overline{P}$ and $\overline{y} \neq \overline{0}$. Hence $\overline{K} \cap \overline{P} \neq \overline{0}$. This shows that $\overline{P}$ is an essential submodule of $\overline{M}$. Conversely let $a \in R$, such that there exists a submodule P of M containing N with the property that $Pa \subseteq N$ and $\overline{P}$ is essential in $\overline{M}$. We then show that $a \in \mathfrak{R}_1(N)$. Let $x \in M$, $x \notin N$. Let K be the submodule of M generated by x. Then $(K + N) - N$ is a non-zero submodule of $\overline{M}$. Thus $\overline{P} \cap [(K + N) - N] \neq \overline{0}$. Consequently we can choose $y \in K$ such that $\overline{y} \neq \overline{0}$ and $\overline{y} \in P$. Then $yR^*a \subseteq Pa \subseteq N$, $y \notin N$. Hence $a \in \mathfrak{R}_1(N)$.

Now rad M denotes $\mathfrak{R}_1(0)$. By taking $N = (0)$ in Proposition 3.1 we conclude

Proposition 3.2. *If* $a \in R$, *then* $a \in \operatorname{rad} M$ *if and only if* $Pa = 0$ *for some essential submodule* P *of* M.

From Proposition 3.2 we get

Proposition 3.3. *If* M *be any* R*-module which does not contain any proper submodule, then* $\operatorname{rad} M = M^r$, *where* M^r *is the right annihilator ideal of* M.

If we regard a ring R as a right R-module, and a right ideal A of R as a R-submodule of R, we get the following

Proposition 3.4. *Let* A *be any right ideal of* R. *If* $a \in R$, *then* $a \in \mathfrak{R}_1(A)$ *if and only if there exists a right ideal* E *of* R *containing* A *such that* $Ea \subseteq A$ *and* $\overline{E} = E - A$ *is an essential* R*-submodule of the difference module* $\overline{R} = R - A$.

Proposition 3.5. *Let* A *be any two sided ideal of* R. *If* $a \in R$, *then* $a \in \mathfrak{R}_2(A)$ *if and only if there exists a two sided ideal* T *of* R *containing* A *such that the ideal* $\overline{T} = T/A$ *is essential in the family of all two sided ideals of the factor ring* $\overline{R} = R/A$ *and* $Ta \subseteq A$.

Proof. Let $a \in \mathfrak{R}_2(A)$. Let $T = \{x \in R \mid xR^*a \subseteq A\}$. T is a two sided ideal of R containing A. To show that T is essential in the family of all two sided ideals of $\overline{R}$, let $\overline{S} = S/A$ be any non-zero two sided ideal of $\overline{R}$. We show that $\overline{S} \cap \overline{T} \neq \bar{0}$. Let $\bar{x}\ (\neq \bar{0}) \in \overline{S}$, then $x \notin A$, and there exists $y \in (x)_t$ such that $y \notin A$ and $yR^*a \subseteq A$. Consequently $\bar{y} \in \overline{S} \cap \overline{T}$ and $\bar{y} \neq \bar{0}$. Thus $\overline{S} \cap \overline{T} \neq \bar{0}$ and $\overline{T}$ is an essential two sided ideal of $\overline{R}$. Clearly $Ta \subseteq A$. Conversely let $a \in R$, for which there exists two sided ideal T of R containing A such that $\overline{T} = T/A$ is an essential two sided ideal of $\overline{R} = R/A$ and $Ta \subseteq A$. Let $x \in R$ with $x \notin A$. Then $[(x)_t + A]/A$ is a non-zero two sided ideal of $\overline{R}$. Consequently $\overline{T} \cap [(x)_t + A]/A \neq \bar{0}$. Thus we choose an element $y \in (x)_t$ such that $\bar{y}(\neq \bar{0}) \in \overline{T} \cap [(x)_t + A]/A$. Then $y \in T$ and $yR^*a \subseteq Ta \subseteq A$. Hence $a \in \mathfrak{R}_2(A)$.

By taking $A = (0)$ we get from Proposition 3.5 the following

Proposition 3.6. *If* $a \in R$, *then* $a \in \operatorname{Rad} R$ *if and only if* $Ta = 0$ *for some essential two sided ideal* T.

4. – In this section we take a ring R with the following properties:

(P_1) R has zero right singular ideal;

(P_2) every non-zero right ideal of R contains a uniform right ideal.

Because of (P_2) R contains a direct sum $\sum_{i\in I}\oplus U_i$ of uniform right ideals of R such that this sum is an essential right ideal of R.

Lemma 4.1. *It is* $\operatorname{rad} R = \bigcap_{i\in I} \operatorname{rad} U_i$.

Proof. Let $a\in \operatorname{rad} R$. By Propositions 3.2 and 3.4, there exists an essential right ideal E of R, such that $Ea=0$. For each i, let $U_i' = U_i\cap E$. Then $U_i' \subset' U_i$ and $U_i' a = 0$. Consequently by Proposition 3.2 $a\in \operatorname{rad} U_i$ and $a\in \in \bigcap_{i\in I} \operatorname{rad} U_i$. Conversely let $b\in \bigcap_{i\in I} \operatorname{rad} U_i$. Thus $b\in \operatorname{rad} U_i$ for every i. Hence there exists a right ideal V_i of R such that $V_i\subset' U_i$ and $V_i b = 0$. If $E = \sum_{i\in I}\oplus V_i$, then $E\subset' R$ and $Eb=0$. Again by Proposition 3.2 $a\in \operatorname{rad} R$. Hence the lemma follows.

Let $\mathfrak{F}$ be the family of all uniform right ideals of R. Define a relation $\sim$ on $\mathfrak{F}$ as follows: if $B_1, B_2\in\mathfrak{F}$, then $B_1\sim B_2$ if and only if $\operatorname{rad} B_1 = \operatorname{rad} B_2$. This is obviously an equivalence relation on $\mathfrak{F}$. For each $B\in\mathfrak{F}$, let S_B denotes the sum of all uniform right ideals which are equivalent to B. Since R has zero right singular ideal, thus for any $x\in R$, and uniform right ideal B' of R, either $xB'=0$ or $xB\cong B'$. This shows that S_B is a two sided ideal of R.

Lemma 4.2. *If B' be any uniform right ideal of R contained in S_B, then $B'\sim B$.*

Proof. From the definition of S_B, we find that S_B contains a direct sum $\sum_{j\in J}\oplus B_j$, of uniform right ideal B_j, such that $B\sim B_j$ and $\sum_{j\in J}\oplus B_j\subset' S_B$. Then $B'\cap \sum_{j\in J}\oplus B_j \neq 0$. Let $B'' = B'\cap\sum_{j\in J}\oplus B_j$. Then B'' is a uniform right ideal contained in B'. Now $B''\subseteq \sum_{j\in J}\oplus B_j$. Consequently B'' has non-zero natural projection into at least one of the B_j, say $B_{j'}$. If $\tau_{j'}\colon B''\to B_{j'}$ be a non-zero projection, then $\tau_{j'}$ is a R-monomorphism, since R has zero right singular ideal and B'' is a uniform right ideal. Consequently $B''\cong\tau_{j'}(B'')$ and $\operatorname{rad} B'' = = \operatorname{rad}\tau_{j'}(B'')$. It can be easily proved that for any R-module M, if N be any essential submodule of M, then $\operatorname{rad} M = \operatorname{rad} N$. Now $B''\subset' B'$, and $\tau_{j'}(B'')\subset' B_j'$, thus $\operatorname{rad} B'' = \operatorname{rad} B'$, $\operatorname{rad}\tau_{j'}(B'') = \operatorname{rad} B_{j'}$. Hence $\operatorname{rad} B' = = \operatorname{rad} B_{j'} = \operatorname{rad} B$, and $B'\sim B$. Hence the result follows.

Now we are in a position to prove:

Theorem 1. $\operatorname{Rad} R = \operatorname{rad} R$, *and* $\operatorname{Cl}(\operatorname{rad} R)$ *is semiprime ideal.*

Proof. Obviously $\operatorname{rad} R\subseteq \operatorname{Rad} R$. Now let $a\in\operatorname{Rad} R$, then by Proposition 3.6 for some essential two sided ideal T of R, $Ta = 0$. Now by Proposition 4.1 $\operatorname{rad} R = \bigcap_{i \in I} \operatorname{rad} U_i$. Consider S_{U_i}. It is a non-zero two sided ideal of R. Thus, $S_{U_i} \cap T \neq 0$ and there exists a uniform right ideal $V_i \subset S_{U_i} \cap T$. Then $V_i a = 0$ and $a \in \operatorname{rad} V_i$. But by Proposition 4.2 $V_i \sim U_i$, and so $a \in \operatorname{rad} U_i$. Hence $a \in \bigcap_{i \in I} \operatorname{rad} U_i = \operatorname{rad} R$. Therefore $\operatorname{Rad} R = \operatorname{rad} R$.

Now to prove that Cl (rad R) is a semiprime ideal, let A be any right ideal of R such that $A^2 \subseteq$ Cl (rad R). Let $A \nsubseteq$ Cl (rad R), then there exists a non-zero right ideal I contained in A such that $I \cap$ Cl (rad R) $= 0$. Consequently $I^2 = 0$, since $I^2 \subseteq$ Cl (rad R) and $I^2 \subseteq I$. Let $K = I + RI$. This K is a two sided ideal of R such that $K^2 = 0$. Let a right ideal J of R be complement of K in R. Then $K + J \subset' R$ and $J.K = 0$. Then $(J + K).I = 0$ and $I \subseteq \operatorname{rad} R$. This is a contradiction. Hence $A \subseteq$ Cl (rad R), and Cl (rad R) is a semiprime ideal.

Theorem 2. *For any two sided ideal A of R belonging to $L^s(R)$, and* $\mathrm{Cl}(\mathfrak{R}_1(A))$ *is a semiprime ideal of R.*

Proof. Let $\bar{R} = R/A$. By using Propositions 3.2 and 3.4 it can be easily proved that $a \in \mathfrak{R}_1(A)$ if and only if $\bar{a} \in \mathfrak{R}_1(\bar{0})$. Similarly Proposition 3.5 will give that $a \in \mathfrak{R}_2(A)$ if and only if $\bar{a} \in \mathfrak{R}_2(\bar{0})$. Consequently $\mathfrak{R}_2(A) = \mathfrak{R}_1(A)$ if and only if $\mathfrak{R}_1(\bar{0}) = \mathfrak{R}_2(\bar{0})$.

From the fact that R satisfies (P_1) and (P_2) and $A \in L^s(R)$ it can be easily proved that $\bar{R}$ has zero right singular ideal and every non-zero right ideal of R contains a uniform right ideal. Consequently $\mathfrak{R}_1(\bar{0}) = \mathfrak{R}_2(\bar{0})$, by Theorem 1. Hence $\mathfrak{R}_1(A) = \mathfrak{R}_2(A)$. Further $\mathrm{Cl}(\mathfrak{R}_1(A))$ is the pre-image of the $\mathrm{Cl}(\mathfrak{R}_1(\bar{0}))$ in R. But $\mathrm{Cl}(\mathfrak{R}_1(\bar{0}))$ is a semiprime ideal by Theorem 1, hence $\mathrm{Cl}(\mathfrak{R}_1(A))$ is also a semiprime ideal of R.

Let R be any ring such that every non-zero right ideal of R contains a minimal right ideal. Here we don't suppose that the right singular ideal of R is zero. It can be proved on similar lines that the Lemmas 4.1 and 4.2 hold if we replace uniform right ideals by minimal right ideals, we can prove the following

Theorem 3. *If R be any ring in which every non-zero right ideal contains a minimal right ideal and R may not have its right singular ideal to be zero, then* $\operatorname{rad} R = \operatorname{Rad} R$.

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References.

[1] L. Lesieur et R. Croisot, *Théorie Noethérienne des anneaux, des demi-groupes et des modules dans le cas non commutatif* (II), Math. Ann. **134** (1957-58), 458-476.

[2] A. W. Goldie, *Torsion-free modules and rings*, J. Algebra **1** (1964), 268-287.

[3] R. E. Johnson, *Quotient rings of rings with zero singular ideals*, Pacific J. Math. **11** (1961), 1385-1392.

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