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Appell's Double Hypergeometric Function as a Generating Function of the Jacobi Polynomial. (**)

1. – Generating functions play a large role in the study of polynomial sets. In section 2 of this paper a generating function for the JACOBI polynomial is obtained. This generating function involves APPELL's second hypergeometric function of two variables. This result is similar to the result given in [2] for APPELL's function F_4. In section 3, by an application to the limiting process to the result of section 2, a generating function for LAGUERRE polynomials is derived. This involves confluent hypergeometric function of two variables.

2. – Definitions and results used:

APPELL's function of two variables is defined ([1], p. 224) by the series

$$F_2(a;\ b,\ b';\ c,\ c';\ x,\ y) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}\,(b)_k\,(b')_n\,x^k\,y^n}{n!\ k!\ (c)_k\,(c')_n}, \tag{2.1}$$

where $(a)_r = a(a+1)\ldots(a+r-1)$, $(a)_0 = 1$ and the double series converges in the region $|x| + |y| < 1$. If we replace y by y/b' and take the limit as $b' \to \infty$, we obtain the confluent series ([1], p. 225):

$$\psi_1(a,\ b;\ c,\ c';\ x,\ y) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}\,(b)_k\,x^k\,y^n}{k!\ n!\ (c)_k\,(c')_n} \tag{2.2}$$

valid for $|x| + |y| < 1$.

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The JACOBI polynomial can be defined by the relation

$$(2.3) \qquad P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1\left(-n,\ 1+\alpha+\beta+n;\ 1+\alpha;\ \frac{1-x}{2}\right),$$

and the LAGUERRE polynomial can be defined as

$$(2.4) \qquad L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_1F_1(-n;\ 1+\alpha;\ x)\,.$$

We shall need the transformation ([1], p. 240)

$$(2.5) \qquad F_2(\alpha;\beta,\beta';\ \gamma,\gamma';\ x,y) = (1-y)^{-\alpha} F_2\left(\alpha;\beta,\gamma'-\beta';\ \gamma,\gamma';\ \frac{x}{1-y},\ \frac{-y}{1-y}\right).$$

3. – The generating relation for the JACOBI polynomial to be proved is

$$(3.1) \qquad \left(\frac{1+x}{2}\right)^{-\alpha-\beta-1} F_2\left(1+\alpha;\ 1+\alpha+\beta,\ c'-b';\ 1+\alpha,\ c';\ \frac{x-1}{x+1},\ y\right) =$$

$$= \sum_{n=0}^{\infty} \frac{(c'-b')_n\, y^n}{(c')_n}\, P_n^{(\alpha,\,\beta-n)}(x)$$

valid for $|y| + |(1-x)/(1+x)| < 1$, $\alpha > -1$, $\beta > -1$. For instance the inequality is satisfied in the region $|y| \leqslant \frac{1}{2}$, $\frac{1}{3} < x < 1$.

To prove (3.1) we start with APPELL's function in the form

$$(1-x)^{-b} F_2\left(a;\ b,\ c'-b';\ c,\ c';\ \frac{-x}{1-x},\ y\right).$$

On applying the transformation in (2.5) the above becomes

$$X = (1-x)^{-b}(1-y)^{-a} F_2\left(a;\ b,\ b';\ c,\ c';\ \frac{-x}{(1-x)(1-y)},\ \frac{-y}{1-y}\right) =$$

$$= \sum_{n,k=0}^{\infty} \frac{(-1)^{n+k}(a)_{n+k}(b)_k(b')_n\, x^k y^n}{k!\, n!\, (c)_k\, (c')_n\, (1-x)^{b+k}(1-y)^{a+n+k}}\,.$$

Since

$$(1-z)^{-m} = \sum_{i=0}^{\infty} \frac{(m)_i\, z^i}{i!},$$

$$X = \sum_{n,k,s,i=0}^{\infty} \frac{(-1)^{n+k}\,(a)_{n+k+s}\,(b)_{k+i}\,(b')_n\, x^{k+i}\, y^{n+s}}{k!\; n!\; i!\; s!\; (c)_k\, (c')_n}.$$

Taking the CAUCHY product of the series in « k » and « i » we have

$$X = \sum_{n,k,s=0}^{\infty} \sum_{i=0}^{k} \frac{(-1)^{n+k-i}\,(a)_{n+k-i+s}\,(b)_k\,(b')_n\, x^k\, y^{n+s}}{(k-i)!\; n!\; i!\; s!\; (c)_{k-i}\,(c')_n}.$$

Then, on reversing the inner summation, we have

$$X = \sum_{n,k,s=0}^{\infty} \sum_{i=0}^{k} \frac{(-1)^{n+i}\,(a)_{n+s+i}\,(b)_k\,(b')_n\, x^k\, y^{n+s}}{i!\; (k-i)!\; n!\; s!\; (c)_i\,(c')_n}.$$

Now employing the results $(k-i)! = (-1)^i\, k!\,/\,(-k)_i$ and $(a)_{n+s+i} = (a)_{n+s}\,(a+n+s)_i$ and summing the inner series, we find that

$$X = \sum_{n,k,s=0}^{\infty} {}_2F_1(-k,\; a+n+s;\; c;\; 1)\; \frac{(-1)^n\,(a)_{n+s}\,(b)_k\,(b')_n\, x^k\, y^{n+s}}{k!\; n!\; s!\; (c')_n}.$$

Again taking the CAUCHY product of the series in « n » and « s » we have

$$X = \sum_{n,k=0}^{\infty} \sum_{s=0}^{n} {}_2F_1(-k,\; a+n;\; c;\; 1)\; \frac{(-1)^{n-s}\,(a)_n\,(b)_k\,(b')_{n-s}\, x^k\, y^n}{k!\; s!\; (n-s)!\; (c')_{n-s}}.$$

If k is a positive integer, then ([2], p. 128) ${}_2F_1(-m,\; d+k;\; d;\; 1) = 0$ for $m > k$ and $= (-k)_m/(d)_m$ for $0 \leqslant m \leqslant k$. Thus, if $c = a$, we conclude that

$$X = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{s=0}^{n} \frac{(-n)_k\,(b)_k\,(a)_n\,(-1)^s\,(b')_s\, x^k\, y^n}{k!\; s!\; (n-s)!\; (a)_k\,(c')_s},$$

where we have reversed the order of summation in « s ». On summing the series in « s » and « k », we obtain

$$X = \sum_{n=0}^{\infty} \frac{(a)_n\, y^n}{n!}\; {}_2F_1(-n,\; b;\; a,\; x)\; {}_2F_1(-n,\; b';\; c';\; 1).$$

Then by means of GAUSS's formula ([2], p. 69)

$$ {}_2F_1(-n,\ \alpha;\ \beta,\ 1) = (\beta-\alpha)_n/(\beta)_n\,, $$

and the transformation (2.5), we finally have

$$ (3.2)\qquad (1-x)^{-b}\, F_2\left(a;\ b,\ c'-b';\ a,\ c';\ \frac{-x}{1-x},\ y\right) = $$

$$ = \sum_{n=0}^{\infty} \frac{(a)_n\,(c'-b')_n\, y^n}{n!\,(c')_n}\ {}_2F_1(-n,\ b;\ a;\ x) $$

valid for $|x/(1-x)| + |y| < 1$, c' not a negative integer or zero. Setting $a = 1+\alpha$, $b = 1+\alpha+\beta$, writing $(1-x)/2$ for x and employing the definition of $P_n^{(\alpha,\beta)}(x)$, the result in (.1) is obtained.

4. – Generating function for the LAGUERRE polynomial:

In (3.2) replace x by x/b and take the limit as $b \to \infty$; remembering that

$$ \operatorname*{Lt}_{\beta\to\infty}\ {}_2F_1(\alpha,\ \beta;\ \gamma;\ x/\beta) = {}_1F_1(\alpha;\ \gamma;\ x)\,, $$

we obtain

$$ (4.1)\qquad e^{x}\,\psi_1(a,\ c'-b';\ a,\ c';\ -x,\ y) = \sum_{n=0}^{\infty} \frac{(a)_n\,(c'-b')_n\, y^n}{n!\,(c')_n}\ {}_1F_1(-n;\ a;\ x)\,. $$

Setting $a = 1+\alpha$ and dropping the primes in b', c', we have the generating relation

$$ (4.2)\qquad e^{x}\,\psi_1(1+\alpha,\ c-b;\ 1+\alpha,\ c;\ -x,\ y) = \sum_{n=0}^{\infty} \frac{(c-b)_n\, y^n}{(c)_n}\ L_n^{(\alpha)}(x) $$

valid for $|x| < 1$, $\alpha > -1$, c not negative integer or zero. With $b = 0$ (4.2) reduces to the well known result

$$ (1-y)^{-\alpha-1}\ e^{-xy/(1-y)} = \sum_{n=0}^{\infty} y^n\ L_n^{(\alpha)}(x)\,. $$

References.

1] A. ERDÉLYI, *Higher Trascendental Functions*, Vol. 1, McGraw-Hill, New York 1953.

[2] E. D. RAINVILLE, *Special Functions*, Macmillan, New York 1960.

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