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On Some Properties of an Integral Function $f(z)*g(z)$. (**)

1. - Let

$$f(z)=\sum_{n=0}^{\infty}a_n z^n \tag{1.1}$$

and

$$g(z)=\sum_{n=0}^{\infty}b_n z^n \tag{1.2}$$

be two integral functions. Then

$$f(z)*g(z)=\sum_{n=0}^{\infty}a_n b_n z^n \tag{1.3}$$

will be an integral function. For

$$\liminf_{n\to\infty}|a_n|^{-1/n}=+\infty, \qquad \liminf_{n\to\infty}|b_n|^{-1/n}=+\infty,$$

therefore

$$\liminf_{n\to\infty}|a_n b_n|^{-1/n}=+\infty.$$

Let $f^{(s)}(z)$ be the s-th derivative of $f(z)$:

$$f^{(s)}(z)=\sum_{n=s}^{\infty}n(n-1)\dots(n-s+1)\,a_n z^{n-s}. \tag{1.4}$$

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Similarly,

$$g^{(s)}(z) = \sum_{n=s}^{\infty} n(n-1)\dots(n-s+1)\, b_n\, z^{n-s}. \tag{1.5}$$

Further, let

$$\left.\begin{aligned} M(r,\ s) &= \max_{|z|=r} |\, f^{(s)}(z) * g^{(s)}(z)\,| \\ M^*(r,\ s) &= \max_{|z|=r} |\, (f(z) * g(z))^{(s)}\,| \end{aligned}\right\} \quad (s = 0,\ 1,\ 2,\ \dots),$$

where $(f(z)*g(z))^{(s)}$ is the s–th derivative of $f(z)*g(z)$, i. e. of the series (1.3).

Let $\mu(r,\ s)$ be the maximum term of $f^{(s)}(z)*g^{(s)}(z)$ for $|\,z\,| = r$ and $\nu(r,\ s)$ the rank of this term.

In this paper we have studied some of the growth properties of $f^{(s)}(z)*g^{(s)}(z)$.

2. – We first prove in this section the following

Lemma 1. *If $f(z)*g(z)$ be an integral function of finite order ϱ, then $f^{(s)}(z)*\ g^{(s)}(z)$ is an integral function of finite order ϱ.*

Proof. It is known ([1], p. 9) that a function $F(z) = \sum_{n=0}^{\infty} c_n z^n$ will be an integral function of finite order ϱ, if and only if

$$\liminf_{n\to\infty} \frac{\log(1/|\,c_n\,|)}{n \log n} = \frac{1}{\varrho}.$$

Now, for $\qquad f^{(s)}(z)*g^{(s)}(z) = \sum_{n=s}^{\infty} n^2(n-1)^2\dots(n-s+1)^2\, a_n\, b_n\, z^{n-s},$

$$\liminf_{n\to\infty} \frac{\log\{1/(n^2(n-1)^2\dots(n-s+1)^2\,|\,a_n\, b_n\,|)\}}{n \log n} =$$

$$= \liminf_{n\to\infty} \frac{\log(1/|\,a_n\, b_n\,|)}{n \log n} = \frac{1}{\varrho}.$$

Hence the lemma.

Theorem 1. *If $f(z)$ and $g(z)$ be two integral functions of finite orders ϱ_1 and ϱ_2 respectively, then $f^{(s)}(z)*g^{(s)}(z)$ is an integral function of finite order ϱ*

such that

$$\frac{1}{\varrho} \geqslant \frac{1}{\varrho_1} + \frac{1}{\varrho_2}. \tag{2.1}$$

Proof. Since $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are integral functions of finite orders ϱ_1 and ϱ_2 respectively and $f(z)*g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$, therefore we have ([2], p. 421) the (2.1).

Further, we have proved in Lemma 1 that $f^{(s)}(z)*g^{(s)}(z)$ is an integral function of the same order as $f(z)*g(z)$. Hence the theorem.

3. – In this section we are going to prove another lemma which will be used in the proof of the next theorem.

Lemma 2. *If $f(z)*g(z)$ be an integral function of finite order ϱ and type $T(0 < T < \infty)$, then $f^{(s)}(z)*g^{(s)}(z)$ will be an integral function of order ϱ and type T.*

Proof. In Lemma 1 we have proved that $f^{(s)}(z)*g^{(s)}(z)$ is an integral function and of order ϱ.

It is known ([1], p. 11) that an integral function $F(z) = \sum_{n=0}^{\infty} c_n z^n$ of finite order ϱ will be of type T $(0 < T < \infty)$ if and only if

$$\limsup_{n\to\infty} \left\{\frac{1}{e\varrho} n \mid c_n \mid^{\varrho/n}\right\} = T.$$

Now, for $$f^{(s)}(z)*g^{(s)}(z) = \sum_{n=s}^{\infty} n^2(n-1)^2 \dots (n-s+1)^2 a_n b_n z^{n-s},$$

$$\limsup_{n\to\infty} \left\{\frac{1}{e\varrho} n \mid n^2(n-1)^2 \dots (n-s+1)^2 a_n b_n \mid^{\varrho/n}\right\} =$$

$$= \limsup_{n\to\infty} \left\{\frac{1}{e\varrho} n \mid a_n b_n \mid^{\varrho/n}\right\} = T.$$

Theorem 2. *If* $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $g(z)=\sum_{n=0}^{\infty} b_n z^n$ *be two integral functions of finite orders* ϱ_1 *and* ϱ_2 *and finite types* T_1 *and* T_2 *respectively, then* $f^{(s)}(z)*g^{(s)}(z)$ *will be an integral function of finite order* ϱ *and finite type* T, *such that:*

(i) $$(eT\varrho)^{1/\varrho} \geqslant (eT_1\varrho_1)^{1/\varrho_1}(eT_2\varrho_2)^{1/\varrho_2};$$

(ii) *if* $f(z)$ *and* $g(z)$ *are of regular growth and* $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ *be non-decreasing functions of* n *for* $n>n_0$, *then*

$$(T\varrho)^{1/\varrho}=(T_1\varrho_1)^{1/\varrho_1}(T_2\varrho_2)^{1/\varrho_2}.$$

Proof. (i) Consider

$$n^{1/\varrho}\left[n^2(n-1)^2\dots(n-s+1)^2|a_n b_n|\right]^{1/n} \geqslant$$

$$\geqslant n^{1/\varrho_1+1/\varrho_2}\left[n^2(n-1)^2\dots(n-s+1)^2|a_n b_n|\right]^{1/n}=$$

$$=n^{1/\varrho_1}\left[n(n-1)\dots(n-s+1)|a_n|\right]^{1/n}\cdot$$

$$\cdot n^{1/\varrho_2}\left[n(n-1)\dots(n-s+1)|b_n|\right]^{1/n}.$$

Taking limits of both sides and using Lemma 1 and Lemma 2, we have

$$(eT\varrho)^{1/\varrho}\geqslant(eT_1\varrho_1)^{1/\varrho_1}(eT_2\varrho_2)^{1/\varrho_2}.$$

(ii) If $f(z)$ and $g(z)$ are integral functions of regular growth and $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ be a non-decreasing functions of n for $n>n_0$, then ([3], p. 26)

$$\frac{1}{\varrho}=\frac{1}{\varrho_1}+\frac{1}{\varrho_2}.$$

Consider

$$n^{1/\varrho}\left[n^2(n-1)^2\dots(n-s+1)^2|a_n b_n|\right]^{1/n}=$$

$$=n^{1/\varrho_1+1/\varrho_2}\left[n^2(n-1)^2\dots(n-s+1)^2|a_n b_n|\right]^{1/n}.$$

Proceeding to limits, we have

$$(T\varrho)^{1/\varrho}=(T_1\varrho_1)^{1/\varrho_1}(T_2\varrho_2)^{1/\varrho_2}.$$

4. – Here we obtain inequalities connecting the maximum modulus of $f^{(s)}(z)*g^{(s)}(z)$.

Theorem 3. *It is*

$$\frac{1}{r}\{M(r,0)-|F(0)|\}\leqslant M(r,1)\leqslant\frac{R_1\,M(R,0)}{(R-R_1)(R_1-r)},$$

where

$$F(0)=f(0)*g(0) \qquad \textit{for} \quad r<R_1<R.$$

Proof. We can easily see that

$$f^{(1)}(z)*g^{(1)}(z)=\frac{\mathrm{d}}{\mathrm{d}z}\left[z\,\frac{\mathrm{d}}{\mathrm{d}z}(f*g)\right].$$

Therefore

$$z\,\frac{\mathrm{d}}{\mathrm{d}z}\big(f(z)*g(z)\big)=\int_0^z f^{(1)}(t)*g^{(1)}(t)\,\mathrm{d}t,$$

where the integral may be taken along a straight line.

Choosing z such that

$$\left|z\,\frac{\mathrm{d}}{\mathrm{d}z}\big(f(z)*g(z)\big)\right|=|z|\;M^*(|z|,1),$$

we have

$$M^*(r,1)\leqslant M(r,1).$$

Again

$$f(z)*g(z)=\int_0^z\frac{\mathrm{d}}{\mathrm{d}t}[f(t)*g(t)]\,\mathrm{d}t+f(0)*g(0),$$

therefore

$$M(r,0)\leqslant r\;M^*(r,1)+|f(0)*g(0)|.$$

Hence

$$\frac{M(r,0)-|F(0)|}{r}\leqslant M^*(r,1)\leqslant M(r,1). \tag{4.1}$$

On the otherhand by CAUCHY's integral formula

$$f^{(1)}(z)*g^{(1)}(z)=\frac{\mathrm{d}}{\mathrm{d}z}\left[z\,\frac{\mathrm{d}}{\mathrm{d}z}(f*g)\right]=\frac{1}{2\pi i}\int_C\frac{t}{(t-z)^2}\,\frac{\mathrm{d}}{\mathrm{d}t}(f*g)\,\mathrm{d}t,$$

where C is the circle $|t-z| = R_1 - r$ $(|z| = r < R_1)$.

Choosing z so that $|f^{(1)}(z)*g^{(1)}(z)| = M(r,\ 1)$ and knowing that $|t| < R_1$, we have

$$M(r,\ 1) \leqslant \frac{R_1\, M^*(R_1,\ 1)}{R_1 - r}.$$

Again

$$\frac{\mathrm{d}}{\mathrm{d}z}[f(z)*g(z)] = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\omega)*g(\omega)}{(\omega - z)^2}\,\mathrm{d}\omega\,,$$

where Γ is the circle $|\omega - z| = R - R_1$ $(|z| = R_1 < R)$. Therefore

$$M^*(R_1,\ 1) \leqslant \frac{M(R,\ 0)}{R - R_1}.$$

Hence

$$M(r,\ 1) \leqslant \frac{R_1\, M^*(R_1,\ 1)}{R_1 - r} \leqslant \frac{R_1\, M(R,\ 0)}{(R - R_1)(R_1 - r)}. \tag{4.2}$$

Combining (4.1) and (4.2), the result follows.

Remark. Taking $R = 4r$ and $R_1 = 2r$ we can easily see that the order of $f(z)*g(z)$ and $f^{(1)}(z)*g^{(1)}(z)$ are same.

Theorem 4. *If $f(z)$ and $g(z)$ be two integral functions given by* (1.1) *and* (1.2) *of finite orders ϱ_1 and ϱ_2 respectively, then*

$$M(r,\ s+1) < M(r,\ s)\ r^{3\varrho - 1 + \varepsilon}\,,$$

where ϱ is the order of $f(z)*g(z)$ and for sufficiently large values of r and $\varepsilon > 0$.

Proof. We know from Theorem 1 that if $f(z)$ and $g(z)$ are integral functions of finite orders ϱ_1 and ϱ_2 respectively, then $f^{(s)}(z)*g^{(s)}(z)$ $(s = 0,\ 1,\ 2,\ \ldots)$ is also an integral function of finite order ϱ such that

$$\frac{1}{\varrho} \geqslant \frac{1}{\varrho_1} + \frac{1}{\varrho_2}.$$

Since $f(z)*g(z)$ is an integral function of finite order ϱ, therefore from a known result ([4], p. 34) we obtain

$$\mu(r,\ s) < M(r,\ s) < \mu(r,\ s)\ r^{\varrho+\varepsilon}\,,$$

where $\mu(r, s)$ is the maximum term of $f^{(s)}(z)*g^{(s)}(z)$ and $\nu(r, s)$ rank of this term.

Now for $f^{(s)}(z)*g^{(s)}(z) = \sum_{n=s}^{\infty} n^2(n-1)^2 \dots (n-s+1)^2 a_n b_n z^{n-s}$,

we have

$$\mu(r, s) = [\nu(r, s)\{\nu(r, s)-1\}\dots\{\nu(r, s)-s+1\}]^2 \,|a_{\nu(r,s)} b_{\nu(r,s)}|\, r^{\nu(r,s)-s},$$

$$\mu(r, s+1) = [\nu(r, s+1)\{\nu(r, s+1)-1\}\dots\{\nu(r, s+1)-s\}]^2 \cdot$$
$$\cdot\, |a_{\nu(r,s+1)} b_{\nu(r,s+1)}|\, r^{\nu(r,s+1)-s}.$$

Therefore

$$\mu(r, s+1) \leqslant \frac{\mu(r, s)}{r}[\nu(r, s+1)-s]^2 \leqslant \frac{\mu(r, s)}{r}[\nu(r, s+1)]^2. \tag{4.3}$$

Hence

$$M(r, s+1) < \mu(r, s+1)\, r^{\varrho+\varepsilon}, \qquad \text{for } \varepsilon > 0,$$
$$\leqslant \mu(r, s)\, r^{3\varrho+\varepsilon-1}, \qquad \text{for large } r,$$
$$\leqslant M(r, s)\, r^{3\varrho+\varepsilon-1}, \qquad \text{for large } r.$$

Remarks:

(1) For large r and $0 < \varrho < 1/3$ it follows from the above theorem

$$M(r, 0) > M(r, 1) > \dots > M(r, s) > \dots .$$

(2) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are two integral functions of same order ϱ_1 and of regular growth and $|a_n/a_{n+1}|$, $|b_n/b_{n+1}|$ be non-decreasing sequences, then

$$M(r, s+1) < M(r, s)\, r^{(3/2)\varrho_1+\varepsilon-1}.$$

Since from Theorem 1 we have $1/\varrho = 2/\varrho_1$, therefore from Theorem 4 the result follows.

(3) If $\varrho' = \max(\varrho_1, \varrho_2)$, then

$$M(r, s+1) < M(r, s)\ r^{(3/2)\varrho'+\varepsilon-1}.$$

Since from Theorem 1 we have $1/\varrho \geqslant (1/\varrho_1) + (1/\varrho_2)$, therefore $1/\varrho \geqslant 2/\varrho'$, and the result follows from Theorem 4.

(4) If ϱ_1 and ϱ_2 satisfy the following conditions $0 < \varrho_1 < \frac{1}{2}$, $0 < \varrho_2 < \frac{1}{2}$, then

$$M(r, 0) > M(r, 1) > \dots > M(r, s) > \dots .$$

Since $1/\varrho \geqslant (1/\varrho_1) + (1/\varrho_2)$, therefore $\varrho < \frac{1}{4}$. Hence from Remark 1 the result follows.

5. – Next, we obtain inequalities involving the maximum term and its rank of

$$f^{(s)}(z) * g^{(s)}(z) = \sum_{n=s}^{\infty} n^2(n-1)^2 \dots (n-s+1)^2\, a_n\, b_n\, z^{n-s}.$$

Theorem 5. *If $\nu(r, s)$, $\nu(r, s+1)$ denote the rank of the maximum term of $f^{(s)}(z) * g^{(s)}(z)$ and $f^{(s+1)}(z) * g^{(s+1)}(z)$ respectively for $|z| = r$, then*

$$\{\nu(r, s) - s\}^2 \leqslant r\, \frac{\mu(r, s+1)}{\mu(r, s)} \leqslant \{\nu(r, s+1) - s\}^2 . \tag{5.1}$$

Proof. We have from (4.3)

$$r\ \frac{\mu(r, s+1)}{\mu(r, s)} \leqslant \{\nu(r, s+1) - s\}^2 .$$

Also

$$\mu(r, s+1) = [\nu(r, s+1)\{\nu(r, s+1) - 1\} \dots \{\nu(r, s+1) - s\}]^2\, |a_{\nu(r,s+1)} b_{\nu(r,s+1)}|\, r^{\nu(r,s+1)-s-1}$$

$$\geqslant [\nu(r, s)\{\nu(r, s) - 1\} \dots \{\nu(r, s) - s\}]^2\, |a_{\nu(r,s)}\, b_{\nu(r,s)}|\, r^{\nu(r,s)-s-1}$$

$$= \{\nu(r, s) - s\}^2\, \mu(r, s)/r ,$$

which leads to the required result.

Applications:

$$\limsup_{r\to\infty} \frac{r\ \mu(r, s+1)/\mu(r, s)}{\log r} = 2\varrho , \tag{i}$$

where ϱ is the finite order of $f(z)*g(z)$.

From the second inequality of (5.1) we have

$$\log\left\{r\,\frac{\mu(r,\ s+1)}{\mu(r,\ s)}\right\} \leqslant 2\log\{\nu(r\ s+1)-s\}$$

and, using Lemma 1 and $\limsup\limits_{r\to\infty}\dfrac{\log\nu(r,\ s+1)}{\log r}=\varrho$, we get

(5.2) $$\limsup_{r\to\infty}\ \frac{\log\{r\,\mu(r,\ s+1)/\mu(r,s)\}}{\log r} \leqslant 2\varrho\,.$$

Similarly, from the first inequality of (5.1) we can deduce

(5.3) $$\limsup_{r\to\infty}\frac{\log\{r\,\mu(r,\ s+1)/\mu(r,\ s)\}}{\log r} \geqslant 2\varrho\,.$$

Combining (5.2) and (5.3) the result follows:

(ii) $$\liminf_{r\to\infty}\frac{\log\{r\,\mu(r,\ s+1)/\mu(r,\ s)\}}{\log r} = 2\lambda\,,$$

this follows from the inequalities in (5.1).

Theorem 6. *If $\nu(r,\ s)$ and $\nu(r,\ s+1)$ denote the rank of the maximum terms $\mu(r,\ s)$ and $\mu(r,\ s+1)$ of $f^{(s)}(z)*g^{(s)}(z)$ and $f^{(s+1)}(z)*g^{(s+1)}(z)$ respectively for $|z|=r$, then, at the points of existence of $\mu'(r,\ s)$, $r>0$,*

$$\frac{\nu(r,\ s)-s}{\nu(r,\ s+1)-s} \leqslant \mu'(r,\ s)\left\{\frac{r}{\mu(r,\ s)\,\mu(r,\ s+1)}\right\}^{1/2} \leqslant 1.$$

Proof. We have

(5.4) $$\log\mu(r,\ s) = \log\mu(r_0\,,\ s) + \int\limits_{r_0}^{r}\frac{\nu(t,\ s)-s}{t}\,dt\,.$$

Differentiating we get

(5.5) $$\frac{\mu'(r,\ s)}{\mu\,(r,\ s)} = \frac{\nu(r,\ s)-s}{r}$$

at the points of existence of $\mu'(r,\ s)$.

Again from (5.1)

$$\nu(r, s)-s \leqslant \left\{\frac{r\,\mu(r, s+1)}{\mu(r, s)}\right\}^{1/2} \leqslant \nu(r, s+1)-s\,. \tag{5.6}$$

Hence from (5.6) and (5.5) we find

$$\frac{\mu'(r, s)}{\{\mu(r, s)\}^{1/2}} \leqslant \left\{\frac{\mu(r, s+1)}{r}\right\}^{1/2}. \tag{5.7}$$

Again, using (5.5) and (5.7), we have

$$\frac{\nu(r, s)-s}{\nu(r, s+1)-s} \leqslant \mu'(r, s)\left\{\frac{r}{\mu(r, s)\,\mu(r, s+1)}\right\}^{1/2} \leqslant 1\,.$$

Corollary:

$$\lim_{r\to\infty}\left(\mu'(r, s)\left\{\frac{r}{\mu(r, s)\,\mu(r, s+1)}\right\}^{1/2}\right)=1.$$

This follows from $\nu(r, s) \sim \nu(r, s+1)$.

Theorem 7. *If $f(z)*g(z)$ be an integral function of order ϱ $(0<\varrho<\infty)$ and lower order λ $(0<\lambda<\infty)$, then*

$$\limsup_{r\to\infty}\frac{\log\mu(r, s)}{\nu(r, s)\log r} \leqslant 1-\frac{\lambda}{\varrho}.$$

Proof.

$$f^{(s)}(z)*g^{(s)}(z)=\sum_{n=s}^{\infty} n^2(n-1)^2\ldots(n-s+1)^2\,a_n\,b_n\,z^{n-s}\,.$$

Since $f(z)*g(z)$ is an integral function of finite order ϱ, therefore, by Lemma 1, $f^{(s)}(z)*g^{(s)}(z)$ is also an integral function of finite order ϱ and so

$$\frac{1}{\varrho}=\liminf_{n\to\infty}\frac{-\log\{n^2(n-1)^2\ldots(n-s+1)^2\,|\,a_n b_n\,|\}}{n\,\log n}\,.$$

Hence

$$-\log\left\{n^2(n-1)^2\ldots(n-s+1)^2\,|\,a_n\,b_n\,|\right\}>((1/\varrho)-\varepsilon)\,n\log n \qquad \text{for} \quad n>n_0\,.$$

Also

$$\log \mu(r,\ s) = \log [\{\nu(r,\ s)\}^2 \{\nu(r,\ s)-1\}^2 \dots \{\nu(r,\ s)-s+1\}^2 \,|\, a_{\nu(r,s)} b_{\nu(r,s)} |] +$$

$$+\{\nu(r,\ s)-s\}\log r <$$

$$< -\big((1/\varrho)-\varepsilon\big)\, \nu(r,\ s) \log \nu(r,\ s) + \nu(r,\ s) \log r \quad \text{for} \quad r > r_0,$$

using the above inequality, or

$$\frac{\log \mu(r,\ s)}{\nu(r,\ s)\log r} \leqslant 1 - \frac{\log \nu(r,\ s)}{\log r}\left(\frac{1}{\varrho} - \varepsilon\right) \qquad \text{for} \quad r > r_0.$$

Hence

$$\limsup_{r\to\infty} \frac{\log \mu(r,\ s)}{\nu(r,\ s)\log r} \leqslant 1 - \frac{\lambda}{\varrho}.$$

Theorem 8. *If $f(z)$ and $g(z)$ be two integral functions, then*

(i) $$\limsup_{r\to\infty} \frac{\log [r\, \mu'(r,\ s)/\mu'(r,\ s-1)]}{\log r} \geqslant \varrho,$$

(ii) $$\limsup_{r\to\infty} \frac{\log \{r[\mu'(r,\ s)/\mu'(r,\ 0)]^{1/s}\}}{\log r} \geqslant \varrho,$$

where ϱ is the order of $f(z)*g(z)$ and $\mu(r,\ s)$ is the maximum term of $f^{(s)}(z)*g^{(s)}(z)$.

Proof. We have from the second inequality of (5.1) and from (5.5)

(5.8) $$\frac{\nu(r,\ s)-s}{r} < \frac{\mu'(r,\ s)}{\mu'(r,\ s-1)}.$$

Therefore

$$\log\left\{r\, \frac{\mu'(r,\ s)}{\mu'(r,\ s-1)}\right\} > \log[\nu(r,\ s)-s].$$

On proceeding to limits and using the result of Lemma 1, (i) follows. Again giving to s in (5.8) the values 1, 2, ..., s and multiplying together, we obtain

$$\frac{\mu'(r,\ s)}{\mu'(r,\ 0)} > \left\{\frac{\nu(r,\ s)-s}{r}\right\}^s.$$

From this result (ii) follows.

Theorem 9. *If $f(z)$ and $g(z)$ be two integral functions, then*

$$\left(\frac{r_2}{r_1}\right)^{\nu(r_1, s)-s} \leqslant \frac{\mu(r_2, s)}{\mu(r_1, s)} \leqslant \left(\frac{r_2}{r_1}\right)^{\nu(r_2, s)-s}, \qquad 0 < r_1 < r_2 . \tag{5.9}$$

Proof. It is

$$\log \mu(r, s) = \log \mu(r_0, s) + \int_{r_0}^{r} \frac{\nu(x, s) - s}{x} \, dx .$$

From this we can write

$$\log \mu(r_2, s) \leqslant \log \mu(r_1, s) + [\nu(r_2, s) - s] \log \frac{r_2}{r_1}, \tag{5.10}$$

$$\log \mu(r_2, s) \geqslant \log \mu(r_1, s) + [\nu(r_1, s) - s] \log \frac{r_2}{r_1} . \tag{5.11}$$

(5.10) and (5.11) lead to (5.9).

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