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R. P. Singh (*)

On Generalized Truesdell Polynomials. (**)

1. – The classical orthogonal polynomials have a generalized Rodrigues' formula [4]

$$\mathrm{P}_n(x) = \frac{1}{K_n\, w(x)}\, \mathrm{D}^n[w(x)\, X^n] \qquad \left(\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{1.1}$$

where K_n is a constant, X is a polynomial in x whose coefficients are independent of n and $w(x)$ is the weight function. If $\frac{w'}{w}$ is a linear function of x, say $\Phi(x)$, then the operational formulae for these polynomials are given as follows.

Consider

$$\mathrm{D}^n[w(x)\, X^n \cdot f(x)] = \mathrm{D}^{n-1}[w(x)\, X^{n-1} \cdot \left(\Phi(x)\, X + nX' + X\mathrm{D}\right) f(x)],$$

and by iteration it yields

$$\mathrm{D}^n[w(x)\, X^n \cdot f(x)] = \prod_{j=1}^{n} [X(\Phi(x) + \mathrm{D}) + jX']\, f(x),$$

where the product has been taken in the operative sense and the factors do not commute.

Also

$$\mathrm{D}^n\, [w(x)\, X^n \cdot f(x)] = \sum_{k=0}^{n} \binom{n}{k}\, \mathrm{D}^{n-k}\left(w(x)\, X^n\right) \mathrm{D}^k f(x),$$

(*) Indirizzo: Dept. of Maths., Regional College of Education, Bhopal (M. P.), India.

(**) Ricevuto: 14-II-1967.

thus we have

$$(1.2)\qquad \prod_{j=1}^{n}[X\,(\Phi(x)+\mathrm{D})+jX']=\frac{1}{w(x)}\sum_{k=0}^{n}\binom{n}{k}\mathrm{D}^{n-k}\left(w(x)\,X^{n}\right)\mathrm{D}^{k}.$$

In particular, if $f(x)=1$,

$$(1.3)\qquad P_n(x)=\frac{1}{k_n}\prod_{j=1}^{n}[X\,(\Phi(x)+\mathrm{D})+jX'].$$

The operational formulae for HERMITE, LAGUERRE and JACOBI polynomials have been discussed in [1], [2], [9]. It may be remarked here that the method is equally effective for obtaining operational formulae for non-orthogonal polynomials and functions which admit the RODRIGUES' formula. GOULD and HOPPER [6] have discussed the operational relations for generalized HERMITE functions, while CHATTERJEA [3] has discussed them for BESSEL polynomials. These relations are easily derived if we use the above technique.

2. - Generalized Truesdell polynomials.

TRUESDELL polynomials [5] are defined as

$$(2.1)\qquad T_n^{\alpha}(x)=x^{-\alpha}e^{x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}[x^{\alpha}e^{-x}].$$

TOSCANO [10] has also considered the same class of polynomials.

We now define generalized TRUESDELL polynomials by the relation

$$(2.2)\qquad T_n^{\alpha}(x,\,r,\,p)=x^{-\alpha}e^{px^{r}}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}[x^{\alpha}e^{-px^{r}}].$$

We shall first write some relations concerning the operator $x\dfrac{\mathrm{d}}{\mathrm{d}x}=\delta$, which will be useful in our investigations.

Some relations:

$$(2.3)\qquad \delta^{n}(x^{\alpha})=\alpha^{n}x^{\alpha},$$

$$(2.4)\qquad e^{t\delta}(f(x))=f(x\,e^{t}),$$

$$(2.5)\qquad \delta^{n}(u\,v)=\sum_{k=0}^{n}\binom{n}{k}\delta^{n-k}u\,\delta^{k}\,v,$$

$$(2.6)\qquad e^{t\delta}(u\,v)=e^{t\delta}u\cdot e^{t\delta}v.$$

Using (2.4) and (2.6), the generalized relation may be written as

$$e^{t\delta}\,[f_1(x)\,f_2(x)\,\ldots] = f_1\,(x\,e^t)\;\;f_2(x\,e^t)\,\ldots\,, \tag{2.7}$$

$$F(\delta)\,\big(x^\alpha f(x)\big) = x^\alpha\;F(\delta+\alpha)\,f(x)\,, \tag{2.8}$$

$$F(\delta)\,\big(e^{g(x)} f(x)\big) = e^{g(x)}\;F(\delta + x\,g')\,f(x)\,. \tag{2.9}$$

The generalized rule of differentiation for this operator is of the form

$$\delta_x^n f(z(x)) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\,\frac{\mathrm{d}^k}{\mathrm{d}z^k}\,f(z)\,\sum_{j=0}^{k}(-1)^j\binom{k}{j} z^{k-j}\;\;\delta_x^n\,z^j\,. \tag{2.10}$$

For ordinary differentation, see [**8**].

To prove it, consider

$$\delta_x^n\;f(z(x)) = \sum_{k=0}^{n}\;A_k^n(x)\;\frac{\mathrm{d}^k}{\mathrm{d}z^k}\,f(z)\,, \tag{2.11}$$

where the coefficients $A_k^n(x)$ satisfy the recurrence relation

$$A_k^n\,(x) = A_{k-1}^{n-1}\,(x)\;\delta_x\,z + \delta_x\,A_k^{n-1}\,(x)\,, \tag{2.12}$$

and

$$A_0^0(x) = 1\,,\qquad A_k^n(x) = 0 \qquad\qquad \text{whenever } k \leqslant 0 \text{ or } k > n.$$

It is easily verified that (2.12) holds for $n = 1, 2, 3, \ldots$, and assume that it holds for a fixed n. Operating on (2.11) once with respect to δ_x, we find that

$$\delta_x^{n+1} f(z(x)) = \sum_{k=1}^{n+1} A_{k-1}^n(x)\;\;\delta_x\,z\;\frac{\mathrm{d}^k}{\mathrm{d}z^k}\,f(z) + \sum_{k=0}^{n}\delta_x\,A_k^n(x)\;\frac{\mathrm{d}^k}{\mathrm{d}z^k}\,f(z) =$$

$$= \sum_{k=0}^{n+1}\,[A_{k-1}^n(x)\;\;\delta_x\,z + \delta_x\,A_k^n(x)]\;\frac{\mathrm{d}^k}{\mathrm{d}z^k}\,f(z) = \sum_{k=0}^{n+1} A_k^{n+1}(x)\;\frac{\mathrm{d}^k}{\mathrm{d}z^k}\,f(z)\,,$$

which establishes the validity of (2.12).

Again to find the esplicit form of $A_k^n(x)$, we proceed as follows: Consider (being $z = z(x)$)

$$\delta_x^n (z - y)^k]_{y=z} = \sum_{j=0}^{n} A_j^n(x) \frac{\mathrm{d}^j}{\mathrm{d}z^j} (z - y)^k]_{y=z}$$

$$= \sum_{j=0}^{n} A_j^n(x)\, j!\binom{k}{j} (z - y)^{k-j}]_{y=z} = k!\, A_k^n(x)\,.$$

Also

$$\delta_x^n (z - y)^k]_{y=z} = \sum_{j=0}^{k} (-1)^{k+j}\binom{k}{j} y^{k-j}\, \delta_x^n z^j]_{y=z} =$$

$$= (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} z^{k-j}\, \delta_x^n z^j\,.$$

Therefore

$$A_k^n(x) = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} z^{k-j}\, \delta_x^n z^j\,,$$

which completes the proof of (2.10).

The other formulae of interest which are immediate consequences of (2.10) are

(2.13) $$\delta_x^n\left(\frac{1}{f(x)}\right) = \sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} \frac{\delta_x^n (f(x))^j}{(f(x))^{j+1}}\,,$$

(2.14) $$\delta_x^n\left(\frac{u}{v}\right) = \sum_{k=0}^{n} \binom{n}{k} \delta_x^{n-k} u \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} \frac{\delta_x^n (v)^j}{(v)^{j+1}}\,.$$

3. - Some operational formulae.

Consider

$$x^{-\alpha} e^{px^r}\left(x \frac{\mathrm{d}}{\mathrm{d}x}\right)^n (x^\alpha e^{-px^r}\cdot f(x)) =$$

$$= x^{-\alpha} e^{px^r}\left(x \frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1} (x^\alpha e^{-px^r}\cdot(\alpha - p\, r\, x^r + x\,\mathrm{D})\, f(x)),$$

and by iteration, we obtain

$$x^{-\alpha} e^{px^r}\left(x \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(x^\alpha e^{-px^r} \cdot f(x)\right) = (\alpha - p\, r\, x^r + x\,\mathrm{D})^n f(x).$$

Again since

$$x^{-\alpha} e^{px^r}\left(x \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(x^\alpha e^{-px^r} \cdot f(x)\right) =$$

$$= x^{-\alpha} e^{px^r} \sum_{k=0}^{n} \binom{n}{k} \left(x \frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-k} \left(x^\alpha e^{-px^r}\right) \left(x \frac{\mathrm{d}}{\mathrm{d}x}\right)^k f(x),$$

thus we obtain

$$(\alpha - p\, r\, x^r + x\,\mathrm{D})^n = \sum_{k=0}^{n} \binom{n}{k} T_{n-k}^{\alpha}(x, r, p)\, \delta^k. \tag{3.1}$$

In particular, if $f(x) = 1$,

$$(\alpha - p\, r\, x^r + x\,\mathrm{D})^n \cdot 1 = T_k^{\alpha}(x, r, p). \tag{3.2}$$

Let us now define the operator

$$\alpha - p\, r\, x^r + x\,\mathrm{D} = \mathfrak{D}. \tag{3.3}$$

The LEIBNIZ rule of differentiation for this operator admits the form

$$\mathfrak{D}^n(u\, v) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{D}^{n-k} u\; \delta^k v. \tag{3.4}$$

It is clear from (3.1) or (3.4) that

$$\mathfrak{D}^n = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{D}^{n-k}(1)\, \delta^k. \tag{3.5}$$

Again

$$\mathfrak{D}^{n+k} = \mathfrak{D}^k\, \mathfrak{D}^n,$$

therefore

$$T_{n+k}^{\alpha}(x, r, p) = \mathfrak{D}^k\, T_n^{\alpha}(x, r, p) = \mathfrak{D}^n\, T_k^{\alpha}(x, r, p). \tag{3.6}$$

Using (2.5), we obtain

$$\delta^k T_m^\alpha(x,\ r,\ p) = \sum_{j=0}^{k} \binom{k}{j} T_{k-j}^{-\alpha}(x,\ r,\ -p)\, T_{m+j}^{\alpha}(x,\ r,\ p)\,. \tag{3.7}$$

The use of (3.6) and (3.7) suggest that the inverse relation to (3.5) is

$$\delta^k = \sum_{j=0}^{k} \binom{k}{j} T_{k-j}^{-\alpha}(x,\ r,\ -p)\, \mathfrak{D}^j\,. \tag{3.8}$$

4. - Generating function.

Starting with RODRIGUES' formula, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T_n^\alpha(x,\ r,\ p) =$$

$$= x^{-\alpha} e^{px^r} \sum_{n=0}^{\infty} \frac{t^n}{n!} \delta_x^n (x^\alpha e^{-px^r}) = x^{-\alpha} e^{px^r} e^{t\delta}(x^\alpha e^{-px^r})\,.$$

The use of (2.7) shows that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T_n^\alpha(x,\ r,\ p) = \exp\left[\alpha t + p x^r (1 - e^{rt})\right]. \tag{4.1}$$

A little calculation shows that

$$T_n^{\alpha+\beta}(x,\ r,\ p+q) = \sum_{k=0}^{n} \binom{n}{k} T_{n-k}^{\alpha}(x,\ r,\ p)\, T_k^{\beta}(x,\ r,\ q)\,. \tag{4.2}$$

Returning to the operational relation (3.1), we have

$$e^{t\mathfrak{D}} f(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} T_n^\alpha(x,\ r,\ p)\, e^{t\delta} f(x)\,,$$

which with the help of (2.4) and (4.1) yields

$$e^{t\mathfrak{D}} f(x) = \exp\left[\alpha t + p x^r (1 - e^{rt})\right] f(x e^t)\,. \tag{4.3}$$

In case, if $f(x) = 1$

$$e^{t\mathfrak{D}}(1) = \exp\left[\alpha t + p x^r (1 - e^{rt})\right]. \tag{4.4}$$

The choice of $f(x) = e^{px^r}$ yields

$$e^{t\mathfrak{D}}\,(e^{px^r}) = \exp\,[\alpha\, t + p\, x^r]\,. \tag{4.5}$$

We also have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, T_{n+m}^{\alpha}(x,\; r,\; p) = \exp[\alpha\, t + p\, x^r (1 - e^{rt})]\; T_m^{\alpha}(x\, e^t,\; r,\; p)\,. \tag{4.6}$$

Further, by combining the relation (3.1) and (3.8), we have the transitory relation

$$\mathfrak{D}_{\alpha,p}^{n} = \sum_{k=0}^{n} \binom{n}{k} T_{n-k}^{\alpha}\,(x,\; r,\; p) \sum_{j=0}^{k} \binom{k}{j} T_{k-j}^{-\beta}\,(x,\; r,\; -q)\; \mathfrak{D}_{\beta,q}^{j}$$

$$= \sum_{j=0}^{n} \mathfrak{D}_{\beta,q}^{j} \sum_{k=j}^{n} \binom{n}{k} \binom{k}{j}\; T_{n-k}^{\alpha}\,(x,\; r,\; p)\; T_{k-j}^{-\beta}\,(x,\; r,\; -q)\,.$$

Using the relation (4.2), it reduces to the form

$$\mathfrak{D}_{\alpha,p}^{n} = \sum_{j=0}^{n} \binom{n}{j}\; T_{n-j}^{\alpha-\beta}\,(x;\; r,\; p-q)\; \mathfrak{D}_{\beta,q}^{j}\,. \tag{4.7}$$

The above relation is analogous to that of GOULD and HOPPER [6].

5. - Expansion of the generalized polynomials.

The use of the relation (2.10) helps us in obtaining the explicit form of these polynomials. Indeed we have

$$T_n^{\alpha}(x,\, r,\, p) = e^{px^r} \sum_{s=0}^{n} \binom{n}{s}\, \alpha^{n-s}\, \delta_x^s e^{-p\, x^r}$$

$$= \sum_{s=0}^{n} \binom{n}{s}\, \alpha^{n-s} \sum_{k=0}^{s} \frac{p^k\, x^{rk}}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j}\, (r\, j)^s$$

$$= \sum_{k=0}^{n} \frac{p^k\, x^{rk}}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{s=k}^{n} \binom{n}{s}\, \alpha^{n-s}\, (r\, j)^s$$

$$= \sum_{k=0}^{n} \frac{p^k\, x^{rk}}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{s=0}^{n} \binom{n}{s}\, \alpha^{n-s}\, (r\, j)^s\,,$$

since

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{s=0}^{k-1} \binom{n}{s} \alpha^{n-s} (r j)^s = 0 ,$$

being the k-th difference of a polynomial of degree $k-1$, and such a difference is zero whenever the order of the difference exceeds the degree of the polynomial. Therefore

$$(5.1) \qquad T_n^{\alpha}(x, r, p) = \sum_{k=0}^{n} \frac{p^k x^{rk}}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\alpha + r j)^n .$$

Further, let $E^n f(x) = f(x + r n)$, and $E = 1 + \Delta$, then (5.1) may be written as

$$T_n^{\alpha}(x, r, p) = \sum_{k=0}^{n} \frac{p^k x^{rk}}{k!} (1 - E)^k \alpha^n ,$$

or more symmetrically

$$T_n^{\alpha}(x, r, p) = \sum_{k=0}^{n} \frac{(-p x^r \Delta)^k}{k!} \alpha^n = \sum_{k=0}^{\infty} \frac{(-p x^r \Delta)^k}{k!} \alpha^n .$$

Thus we have

$$(5.2) \qquad T_n^{\alpha}(x, r, p) = e^{-p x^r \Delta} \alpha^n .$$

This relation may very well be regarded as the starting point of the present study. The generating function for these polynomials is now easily verified. Indeed we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T_n^{\alpha}(x, r, p) = e^{-p x^r \Delta} e^{\alpha t} .$$

Using the relation $e^{k \Delta} e^{\alpha t} = e^{\alpha t} e^{k (e^{rt} - 1)}$, the above relation reduces to (4.1).

STIRLING numbers and STIRLING polynomials [7] are defind as follows:

$$(5.3) \qquad S(n, k) = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n = \frac{1}{k!} \Delta^k 0^n ,$$

$$(5.4) \qquad A_n(x) = \sum_{k=0}^{n} S(n, k) x^k .$$

Let

$$(5.5) \qquad T_n^{\alpha}(x, r, -p) = \sum_{k=0}^{n} S^{\alpha}(n, k, r) p^k x^{rk} ,$$

so that from (5.1)

$$S^{\alpha}(n,\ k,\ r) = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\alpha + r\, j)^n. \tag{5.6}$$

Then clearly

$$T_n(x,\ 1,\ -1) = A_n(x)\,, \tag{5.7}$$

$$S(n,\ k,\ 1) = S(n,\ k)\,. \tag{5.8}$$

Thus $S^{\alpha}(n, k, r)$ and $T_n^{\alpha}(x, r, -p)$ may very well be regarded as the generalized STIRLING numbers and STIRLING polynomials. A little calculation shows that

$$S^{\alpha}(n+1,\ k,\ r) = r\, S^{\alpha}(n,\ k-1,\ r) + (\alpha + r\, k)\, S^{\alpha}(n,\ k,\ r)\,. \tag{5.è}$$

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