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Absolute Summability Factors of a Fourier Series. (**)

§ 1.

1.1. - Definition. Let $\sum a_n$ be a given infinite series and S_n^α be the n-th CESÀRO mean of order α $(\alpha > -1)$ of the sequence $\{S_n\}$, where S_n is the partial sum of the given series. The series $\sum a_n$ is said to be absolutely summable (C, α) or summable $|C, \alpha|$, if the series

$$\sum_{n=1}^{\infty} | S_n^\alpha - S_{n-1}^\alpha |$$

is convergent.

A sequence $\{\lambda_n\}$ is said to be convex when $\Delta^2\lambda_n \geqslant 0$ $(n = 1, 2, ...)$, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

1.2. - Let $f(t)$ be a periodic function with period 2π and integrable in the sense of LEBESGUE over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the FOURIER series of $f(t)$ be zero, so that

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) .$$

We write

$$\varphi(t) = \varphi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}.$$

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1.3. - Introduction. Recently PATI has proved the following result:

Theorem A [7]. If $\{\lambda_n\}$ is a convex sequence such that $\sum_{n=1}^{\infty} n^{-1}\lambda_n(\log n)^{1/2} < \infty$, then $\sum_{n=1}^{\infty} \lambda_n A_n(t)$ is summable $|C, 1|$ at every point $t = x$, at which

$$\int_0^t |\varphi(u)|\, du = o(t).$$

SHENG LIU has generalised this result in the following form:

Theorem B [5]. If $\{\lambda_n\}$ is a convex sequence such that

$$\sum_{n=1}^{\infty} n^{-1}\lambda_n (\log n)^{(1/2)(1-\alpha)} < \infty \qquad (0 \leqslant \alpha < 1),$$

then the series $\sum_{n=1}^{\infty} \lambda_n A_n(t)$ is summable $|C, 1|$ at every point $t = x$, at which

$$\Phi(t) = \int_0^t |\varphi(u)|\, du = o\{t/(\log(1/t))^{\alpha}\} \qquad \text{as } t \to 0.$$

The object of this Note is to further extend these results. Infact we prove:

Theorem. *If* $\{\lambda_n\}$ *is a convex sequence such that*

$$\sum n^{-1}\lambda_n (\log n)^{(1/2)-\alpha} \qquad (|\alpha| \leqslant 1/2)$$

is convergent, then the series $\sum_{n=1}^{\infty} \lambda_n A_n(t)/\{n \ (\log n)^{\alpha+\beta-1/2}\}$, $t = x$, *is summable* $|C, \delta|$, $0 < \delta < 1$, *provided that*

$$\Phi(t) = \int_0^t |\varphi(u)|\, du = O\{t(\log(1/t))^{\beta}\}, \qquad \beta \geqslant 0. \tag{1.3.1}$$

§ 2.

We require the following lemmas for the proof of our theorem.

Lemma 1 [2]. If $0 < \delta < 1$, $0 < t \leqslant 2\pi$ and

$$S_n^{\delta}(t) = \sum A_{n-\mu}^{\delta-1}\, \mu \cos(\mu t),$$

then

$$S_n^\delta(t) = \begin{cases} O(n^2) & \text{for } t > 0 \\ O(n\, t^{-\delta}) & \text{for } t > 1/n . \end{cases}$$

Lemma 2 [1]. If $0 < \delta < 1$ and $0 \leqslant m \leqslant n$, then

$$\Big|\sum_{\nu=0}^{m} A_{n-\nu}^{\delta-1}\, a_\nu \Big| \leqslant \max_{0 \leq \mu \leq m} \Big| \sum_{\nu=0}^{\mu} A_{\mu-\nu}^{\delta-1}\, a_\mu \Big| .$$

Lemma 3. *Let* $0 < \delta < 1$ *and* $0 < t \leqslant 2\pi$, *we write*

$$K_n^\delta(t) = \frac{1}{A_n^\delta} \sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} \frac{\lambda_\nu}{(\log \nu)^{\alpha+\beta-(1/2)}}\, \nu^\delta \cos(\nu t) , \qquad \beta \geqslant 0 ,$$

then

$$K_n^\delta(t) = O\left\{n^{-\delta} \sum_{\nu=2}^{n} \nu^{1+\delta} (\log \nu)^{-\alpha-\beta+(1/2)}\, \Delta\lambda_\nu\right\} + O\left\{n\lambda_n/(\log n)^{\alpha+\beta-(1/2)}\right\} \textit{ for } 0 < t \leqslant 1/n$$

and

$$K_n^\delta(t) = O\left\{(n\,t)^{-\delta} \sum_{\nu=2}^{n} \nu^\delta (\log \nu)^{-\alpha-\beta+(1/2)} \Delta\lambda_\nu\right\} + O\left\{t^{-\delta}\lambda_n/(\log n)^{\alpha+\beta-(1/2)}\right\} \quad \textit{for } t > 1/n .$$

Proof. Using Lemmas 1 and 2 and by ABEL's transformation, we have

$$K_n^\delta(t) = \frac{1}{A_n^\delta} \left\{ \sum_{\nu=0}^{n-1} \Delta \left(\frac{\lambda_\nu}{\nu^{1-\delta} (\log \nu)^{\alpha+\beta-(1/2)}} \right) \sum_{\mu=1}^{\nu} A_{n-\mu}^{\delta-1}\, \mu \cos(\mu\, t) \right\} + \frac{S_n^\delta(t)\, \lambda_n}{A_n^\delta n^{1-\delta} (\log n)^{\alpha+\beta-(1/2)}} .$$

For $0 < t \leqslant 1/n$,

$$K_n^\delta(t) = O\left[\frac{1}{A_n^\delta} \sum_{\nu=2}^{n} \Delta \left\{ \frac{\lambda_\nu}{\nu^{1-\delta} (\log \nu)^{\alpha+\beta-(1/2)}} \right\} \nu^2 \right] + O\left[\frac{n^2\, \lambda_n}{A_n^\delta\, n^{1-\delta} (\log n)^{\alpha+\beta-(1/2)}} \right] =$$

$$= O\left[n^{-\delta} \left\{ \sum_{\nu=2}^{n} \frac{\nu^{1+\delta}\, \Delta\lambda_\nu}{(\log \nu)^{\alpha+\beta-(1/2)}} + \sum_{\nu=2}^{n} \frac{\nu^\delta\, \lambda_\nu}{(\log \nu)^{\alpha+\beta-(1/2)}} + \sum_{\nu=2}^{n} \frac{\nu^\delta\, \lambda_\nu}{(\log \nu)^{\alpha+\beta+(1/2)}} \right\} \right] +$$

$$+ O\left[\frac{n\, \lambda_n}{(\log n)^{\alpha+\beta-(1/2)}} \right].$$

Since

$$\sum_{\nu=2}^{n} \frac{\nu^\delta \lambda_\nu}{(\log \nu)^{\alpha+\beta-(1/2)}} = O\left[\sum_{\nu=2}^{n} \Delta\lambda_\nu \sum_{m=2}^{\nu} \frac{m^\delta}{(\log m)^{\alpha+\beta-(1/2)}}\right] + O\left[\lambda_n \sum_{m=2}^{n} \frac{m^\delta}{(\log m)^{\alpha+\beta-(1/2)}}\right],$$

Thus

$$K_n^\delta(t) = O\left\{n^{-\delta} \sum_{\nu=2}^{n} \frac{\nu^{\delta+1} \Delta\lambda_\nu}{(\log \nu)^{\alpha+\beta-(1/2)}}\right\} + O\left\{\frac{n \lambda_\nu}{(\log n)^{\alpha+\beta-(1/2)}}\right\}.$$

Now, for $t > 1/n$,

$$K_n^\delta(t) = O\left\{n^{-\delta} \sum_{\nu=2}^{n} \nu t^{-\delta} \Delta\left(\frac{\lambda_\nu}{\nu^{1-\delta} (\log \nu)^{\alpha+\beta-(1/2)}}\right)\right\} + O\left\{\frac{n^{-\delta} \lambda_n n t^{-\delta}}{n^{1-\delta}(\log n)^{\alpha+\beta-(1/2)}}\right\} =$$

$$= O\left[(n t)^{-\delta}\left\{\sum_{\nu=2}^{n} \frac{\nu \Delta\lambda_\nu}{\nu^{1-\delta}(\log \nu)^{\alpha+\beta-(1/2)}} + \sum_{\nu=2}^{n} \frac{\nu \lambda_\nu}{\nu^{2-\delta}(\log \nu)^{\alpha+\beta-(1/2)}} + \right.\right.$$

$$\left.\left. + \sum_{\nu=2}^{n} \frac{\nu \lambda_\nu}{\nu^{2-\delta}(\log \nu)^{\alpha+\beta+(1/2)}}\right\}\right] + O\left[\frac{t^{-\delta} \lambda_n}{(\log n)^{\alpha+\beta-(1/2)}}\right].$$

Since, we have

$$\sum_{\nu=2}^{n} \frac{\lambda_\nu}{\nu^{1-\delta}(\log \nu)^{\alpha+\beta-(1/2)}} = O\left\{\sum_{\nu=0}^{n} \Delta\lambda_\nu \sum_{\mu=2}^{\nu} \frac{1}{\mu^{1-\delta}(\log \mu)^{\alpha+\beta-(1/2)}}\right\} + O\left\{\lambda_n \sum_{\nu=2}^{n} \frac{1}{\nu^{1-\delta}} \frac{1}{(\log \nu)^{\alpha+\beta-(1/2)}}\right\}$$

$$= O\left\{\sum_{\nu=2}^{n} \frac{\nu^\delta \Delta\lambda_\nu}{(\log \nu)^{\alpha+\beta-(1/2)}}\right\} + O\left\{\frac{\lambda_n n^\delta}{(\log n)^{\alpha+\beta-(1/2)}}\right\}.$$

Thus

$$K_n^\delta(t) = O\left\{(n t)^{-\delta} \sum_{\nu=2}^{n} \frac{\nu^\delta \Delta\lambda_\nu}{(\log \nu)^{\alpha+\beta-(1/2)}}\right\} + O\left\{\frac{\lambda_n t^{-\delta}}{(\log n)^{\alpha+\beta-(1/2)}}\right\}.$$

This completes the proof of the lemma.

Lemma 4 [4]. If

$$\Phi(t) = \int_0^t |\varphi(u)|\, \mathrm{d}u = O\{t(\log(1/t))^\beta\}, \qquad \beta \geqslant 0, \qquad \text{as } t \to 0,$$

then

$$\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t}\, \mathrm{d}t = O\{(\log n)^{\beta+1}\}$$

and

$$\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t^{\delta}}\, dt = O\{(\log n)^{\beta}\} \qquad \text{for } 0 < \delta < 1 .$$

Lemma 5. If $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then

(i) [3] $\{\lambda_n\}$ is non-negative and decreasing, with

$$n\,\Delta\lambda_n = O(1) \quad \text{and} \quad \lambda_n \log n = o(1) \qquad \text{as } n \to \infty;$$

(ii) [6]

$$\sum_{n=1}^{m} \log(n+1)\cdot\Delta\lambda_n = O(1) \qquad \text{as } m \to \infty$$

and

$$n \log n \cdot \Delta\lambda_n = o(1) \qquad \text{as } n \to \infty .$$

§ 3. - Proof of the Theorem.

Let us denote the n-th CESÀRO mean of order δ of the sequence $\left\{\frac{n^{\delta}\lambda_n A_n(x)}{(\log n)^{\alpha+\beta-(1/2)}}\right\}$ by T_n^{δ}. Now

$$T_n^{\delta}(x) = \frac{2}{\pi}\int_0 \varphi(t)\,\frac{1}{A_n^{\delta}}\sum_{\nu=2}^{n}\frac{A_{n-\nu}^{\delta-1}\,\nu^{\delta}\,\lambda_\nu \cos(\nu t)}{(\log \nu)^{\alpha+\beta-(1/2)}}\,dt = \frac{2}{\pi}\left\{\int_0^{1/n} + \int_{1/n}^{\pi}\right\}\varphi(t)\,K_n^{\delta}(t)\,dt,$$

therefore

$$T_n^{\delta}(x) = J_1 + J_2 . \tag{3.1}$$

Using Lemma 3, we write

$$\sum_{n=2}^{m} n^{-1}|J_1| = O\left\{\sum_{n=2}^{m} n^{-1}\int_0^{1/n}\frac{|\varphi(t)|}{n^{\delta}}\sum_{\nu=2}^{n}\frac{\nu^{1+\delta}\,\Delta\lambda_\nu}{(\log \nu)^{\alpha+\beta-(1/2)}}\,dt\right\} +$$

$$+ O\left\{\sum_{n=2}^{m} n^{-1}\int_0^{1/n}\frac{|\varphi(t)|\,n\,\lambda_n}{(\log n)^{\alpha+\beta-(1/2)}}\,dt\right\},$$

therefore

$$\sum_{n=2}^{m} n^{-1} \,|\, J_1 \,| = \sum_{n=2}^{m} n^{-1} \,|\, J_{1,1} + J_{1,2} \,| \,. \tag{3.2}$$

Now using the hypothesis (1.3.1), we have

$$\sum_{n=2}^{m} n^{-1} \,|\, J_{1,1} \,| = O\left\{ \sum_{\nu=2}^{m} \frac{\nu^{1+\delta} \,\Delta\lambda_\nu}{(\log \nu)^{\alpha+\beta-(1/2)}} \sum_{n=\nu}^{m} \frac{(\log n)^\beta}{n^{\delta+2}} \right\}$$

$$= O\left\{ \sum_{\nu=2}^{m} \frac{\nu^{1+\delta} \,\Delta\lambda_\nu}{(\log \nu)^{\alpha+\beta-(1/2)}} \frac{(\log \nu)^\beta}{\nu^{\delta+1}} \right\}$$

$$= O\left\{ \sum_{\nu=2}^{m} \Delta\lambda_\nu \,(\log \nu)^{(1/2)-\alpha} \right\},$$

therefore, by Lemma 5,

$$\sum_{n=2}^{m} n^{-1} \,|\, J_{1,1} \,| < \infty \,. \tag{3.3}$$

Again by hypethesis (1.3.1), we have

$$\sum_{n=2}^{m} n^{-1} \,|\, J_{1,2} \,| = O\left\{ \sum_{n=2}^{m} \frac{\lambda_n}{(\log n)^{\alpha+\beta-(1/2)}} \frac{(\log n)^\beta}{n} \right\} = O\left\{ \sum_{n=2}^{m} n^{-1} \lambda_n \,(\log n)^{(1/2)-\alpha} \right\},$$

therefore, by Lemma 5,

$$\sum_{n=2}^{m} n^{-1} \,|\, J_{1,2} \,| < \infty \,. \tag{3.4}$$

Further

$$\sum_{n=2}^{m} n^{-1} \,|\, J_2 \,| =$$

$$= O\left\{ \sum_{n=2}^{m} n^{-1-\delta} \int_{1/n}^{\pi} \frac{|\,\varphi(t)\,|}{t^\delta} \sum_{\nu=2}^{n} \frac{\nu^\delta \,\Delta\lambda_\nu}{(\log \nu)^{\alpha+\beta-(1/2)}} \,dt \right\} + O\left\{ \sum_{n=2}^{m} n^{-1} \int_{1/n}^{\pi} \frac{|\,\varphi(t\,|}{t^\delta} \frac{\lambda_n}{(\log n)^{\alpha+\beta-(1/2)}} \,dt \right\}$$

$$= O\left\{ \sum_{n=2}^{m} n^{-1-\delta} (\log n)^\beta \sum_{\nu=2}^{n} \frac{\nu^\delta \,\Delta\lambda_\nu}{(\log \nu)^{\alpha+\beta-(1/2)}} \right\} + O\left\{ \sum_{n=2}^{m} n^{-1} (\log n)^\beta \frac{\lambda_n}{(\log n)^{\alpha+\beta-(1/2)}} \right\} =$$

$$= O\left\{\sum_{\nu=2}^{m} \frac{\nu^{\delta}\, \Delta\lambda_{\nu}}{(\log \nu)^{\alpha+\beta-(1/2)}} \sum_{n=\nu}^{m} \frac{(\log n)^{\beta}}{n^{1+\delta}}\right\} + O\left\{\sum_{n=2}^{m} n^{-1}\, \lambda_n\, (\log n)^{(1/2-\alpha)}\right\}$$

$$= O\left\{\sum_{\nu=2}^{m} \frac{\nu^{\delta}\, \Delta\lambda_{\nu}}{(\log \nu)^{\alpha+\beta-(1/2)}} \frac{(\log \nu)^{\beta}}{\nu^{\delta}}\right\} + O(1)$$

$$= O\left\{\sum_{\nu=2}^{m} \Delta\lambda_{\nu}\, (\log \nu)^{(1/2)-\alpha}\right\} + O(1)\,,$$

therefore, by Lemma 5,

$$\sum_{n=2}^{m} n^{-1}\, |\, J_2\, | < \infty\,. \tag{3.5}$$

Collecting (3.3), (3.4) and (3.5), the proof of the Theorem is complete.

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