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Set-valued Additive Functions. (**)

This work is a continuation of [2]. In [2] we established the structure of additive set-valued functions defined on the HAMEL Rational Base Cone of a linear space. In particular, the following theorem was proven:

Let X and Y be linear spaces over the field of rationals and let $C(Y)$ denote the space of all convex sets of the space Y and f_a, e_a $(a \in A)$ be a biorthogonal Hamel Base System in the space X. If $A(x)$ is an additive function from the Hamel Base Cone S into the space $C(Y)$ then $A(x)$ is rationally homogenuous and has the following structure $A(x) = \sum_{a \in A} f_a(x) K_a$ for $x \in S$, where $K_a = A(e_a)$ $(a \in A)$.

In analysis where one is concerned with spaces in which a topology has been defined, it is possible to show explicitely the biorthogonal bases system of such spaces and therefore the representation of additive functions should also be effective. It is therefore natural to investigate additive functions defined on base cones generated by biorthogonal topological base systems. To this end a theorem will be stated which is significant by itself and also shows what kind of results can be expected.

Theorem 1. *Let A be a real-valued additive function defined on the set S of all positive numbers. Then the following conditions are equivalent:*

(a) *A is bounded below in an open interval,*

(b) *A is bounded above in an open interval,*

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(c) A *is bounded in an open interval,*

(d) A *is continuous at a point* $t \in S$,

(e) A *is continuous at every point of* S,

(f) A *is of the form* $A(t) = t\,A\,(1)$ *for all* $t \in S$.

The equivalence of (d), (e) and (f) is well known. The proof given that (a) implies (f) is standard and will be deleted.

The next theorem is analogous to Theorem 1. Let Y denote a reflexive BANACH space and let $C(Y)$ be the space of all non-empty, closed, convex, and bounded sets of the space Y. The space $C(Y)$ forms a semi-linear space under the operation of algebraic addition of sets and algebraic multiplication of a set by a scalar. If V is a neighborhood of zero in a locally convex topology of the space Y then the family of sets $N = \{B: A \subset A + V \text{ and } B \subset A + V\}$ constitutes a base of neighborhoods for the set A in $C(Y)$. This topology is said to be the weak (or strong) topology of $C(Y)$ if it is generated by the weak (or strong) topology of the space Y.

Theorem 2. *Let* A *be an additive function defined on the set* S *of positive reals and have values in the space* $C(Y)$. *Then the following conditions are equivalent:*

(a) A *is bounded in an open interval* (c, d),

(b) A *is continuous at a point* $t \in S$ *in the weak topology of* (CY),

(c) A *is continuous at a point* $t \in S$ *in the strong topology of* (CY),

(d) A *is continuous at every point* $t \in S$ *in the strong topology of* (CY),

(e) A *is of the form* $A(t) = t\,K$, *for all* $t \in S$, *where* $K = A(1) \in C(Y)$.

To prove the Theorem, one makes use of RÅDSTRÖM's Theorem:

Let Y *be any locally convex Hausdorff space and let* $C(Y)$ *denote the set of all compact convex sets of* Y. *Let* $A(t)$ *be an additive function defined on the set of positive reals* S *and with values in* $C(Y)$. *Then there exists a convex compact set* $K \subset C(Y)$ *and an additive function* $f(t)$ *defined on* S *with values in* Y *such that* $A(t) = t\,K + f(t)$ *for all* $t \in S$.

Since Y is reflexive, the members of the space $C(Y)$ are weakly compact. Thus it follows from the RÅDSTRÖM Theorem that there exists an additive vector-valued function f such that $A(t) = f(t) + t\,K$. Weak continuity of the operation of scalar multiplication on the semilinear space $C(Y)$ implies that A is weakly continuous at t_0 if and only if f is weakly (or strongly) continuous at t_0. This establishes the equivalence of (b) and (c). The equivalence of (c), (d) and (e) now follows from the equivalence of (d), (e) and (f) in Theorem 1. Note that $A(1) = f(1) + K$. To prove the equivalence of (a) and (e) one needs only observe that A is bounded on an open interval if and only if f is bounded on an open interval. The equivalence of (a) and (f) in Theorem 1 now imply the assertion.

Remark to Theorem 1 and Theorem 2.

If instead of the function $A(t)$ one considers a function defined for $t \in R^+ = \{t: t \geqslant 0\}$ then the function $A(t)$ will be of the same form, that is, $A(t) = t\,A(1)$ for $t \in R^+$. To show this, it is sufficient to show that $A(0) = \{0\}$, that is, $A(0)$ consists only of one point O. Indeed, one has for $t > 0$.

$$t\;A(1) = A(t) = A(t+0) = A(t) + A(0) = t\;A(1) + A(0)\,.$$

This means that $A(0) \subset t\{A(1) - A(1)\}$ for any $t > 0$. Therefore $\|A(0)\| \leqslant 2t\,\|A(1)\|$ for any $t > 0$. Hence $\|A(0)\| = 0$ and we have that $x \in A(0)$ if and only if $\|x\| = 0$, that is, $A(0) = \{0\}$.

One now proceeds to prove a theorem concerning additive set-valued functions which are defined on the base cone in Banach spaces.

Theorem 3. *Let X be a Banach space with a biorthogonal base system (e_n), (f_n) and let S denote the base cone, that is, $S = \{x \in X: f_n(x) \geqslant 0$ for $n = 1, 2 \ldots\}$. Let Y be a reflexive Banach space and let $C_s(Y) = C(Y)$ be the space of all non-empty, closed, bounded, convex sets of the space Y with its strong topology. Then for every continuous additive function A defined on S and with values in $C(Y)$ there exists sets $K_n \in C(Y)$ such that $A(x) = \sum_{n=1}^{\infty} f_n(x)\,K_n$ for all $x \in S$, the last series being convergent in the strong topology of $C(Y)$.*

To prove the theorem one considers the function $x = t\,e_n$ for positive real t. This function is additive and continuous from the set of positive reals R^+ into S. Therefore its composite with the continuous and additive function A, $B_n(t) = A(t\,e_n)$ for $t \in R^+$ is continuous and additive from R^+ into $C(Y)$. It follows from Theorem 2 that B_n is of the form $A(t\,e_n) = B_n(t) = t\,K_n$ for $t \in R^+$, where $K_n \in C(Y)$. For any point $x \in S$ and $x_n = \sum_{j=1}^{n} f_j(x)\,e_j$, one has:

$$f_k(x_n) = \begin{cases} f_k(x) & \text{if} \quad k \leqslant n \\ 0 & \text{if} \quad k > n\,. \end{cases}$$

Hence $x_n \in S$ and x_n converges to x in the topology of S. Also

$$A(x_n) = A[\sum_{j=1}^{n} f_j(x) \ e_j] = \sum_{j=1}^{n} A[f_j(x) \, e_j] = \sum_{j=1}^{n} f_j(x) \, K_j \, .$$

Therefore it follows from continuity of A that $x_n \rightarrow x$ if and only if $A\,(x_n) \rightarrow A(x)$ and $A(x) = \lim_{n \rightarrow \infty} A(x_n) = \sum_{j=1}^{\infty} f_j(x) \, K_j$. This completes the proof of the theorem.

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