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SURJEET SINGH (*)

# On Pseudo-injective Modules. (**)

## 1. - Introduction.

Let $R$ be any ring. An $R$-module $M$ is said to be a *pseudo-injective* module if for each submodule $N$ of $M$, every $R$-monomorphism of $N$ into $M$ can be extended to an $R$-endomorphism of $M$. The concept of pseudo-injective modules was introduced by the author and S. K. JAIN [7]. Some conditions [6] were determined under which a pseudo-injective module is quasi-injective. In this paper we prove two theorems, one of which states that any pseudo-injective unital module over a principal ideal domain is quasi-injective. Other result is given in Theorem (4.7).

## 2. - Preliminary definitions and notations.

All the rings considered in this paper are supposed to be with unity, and every module is supposed to be unital. For any ring $R$, by an $R$-module we shall always mean a right $R$-module. An $R$-module $M$ is said to be an *injective $R$-module* if for each $R$-module $A$ and for each submodule $B$ of $A$, every $R$-homomorphism of $B$ into $M$ can be extend to an $R$-homomorphism of $A$ into $M$. An $R$-module $M$ is said to be a quasi-injective $R$-module if for each sub-module $N$ of $M$, every $R$-homomorphism of $N$ into $M$, can be extended to an $R$-endomorphism of $M$. The symbols $M^{\Delta}$, $R^{\Delta}$ denote the singular $R$-submodule of $M$, and right singular ideal of $R$ respectively. Let $R$ be a ring with $R^{\Delta}=(0)$. In this case $\hat{R}$ denotes the maximal right quotient ring of $R$ as defined by JOHNSON [3]. It is known that if $Q$ be any ring such that $R \subseteq Q \subseteq \hat{R}$, then $\hat{Q}=\hat{R}$.

(*) Indirizzo: Department of Mathematic, Kirori Mal College, Delhi 7, India.
(**) Ricevuto: 24-XI-1967.

A ring $R$ is said to be a right GOLDIE ring if it has following properties: (i) $R$ satisfies ascending chain condition (a. c. c.) on annihilator right ideals, and (ii) $R$ does not contain any infinite direct sum of right ideals. Let $R$ be a semiprime right GOLDIE ring. GOLDIE [1] proved that $R$ has a classical right quotient ring $S$, which is semisimple artinian. As a consequence $R^{\Delta}=(0)$ and $\widehat{R}$ exist. JOHNSON [5] proved that $\widehat{R}=S$.

3. – Let $M$ be an $R$-module. An element $x \in M$ is said to be *torsion element* if $x\,a=0$ for some regular element $a$ (i. e. an element with is not a zero divisor) otherwise it is said to torsion free element. An $R$-module $M$ is said to be a *torsion module* if every element of $M$ is a torsion element.

(3.1) Lemma. *Let $M$ be a pseudo-injective module over a principal ideal domain (commutative) $R$, such that $M$ is not a torsion module, then $M$ is injective.*

Proof. Since $M$ is not a torsion module, therefore there exist $x \in M$ such that $x\,a \neq 0$ for every $a(\neq 0) \in R$. Let $a(\neq 0) \in R$, $N = x\,a\,R$. Then $N$ is a submodule of $M$ and the mapping

$$\eta: \quad N \rightarrow M$$

such that $\eta(xab)=xb$ ıor every $b \in R$ is a $R$-monomorphism. Let $\xi$ be an endomorphism of $M$ which is an extension of $\eta$. Let $\xi(x)=y$. Then $y\,a=\xi(x\,a)=x$. Thus for each torsion free element $u \in M$ andf or each $a(\neq 0) \in R$, there exist $v \in M$ such that

$$u = v\,a\,. \tag{$*$}$$

Now let $h$ be any torsion element of $M$. Then $x+h$ is torsion free. Thus given $a \in R$, $a \neq 0$, there exist $z \in M$ such that $x+h=z\,a$. Then $h=(z-y)\,a$. This fact along with ($*$) implies that $M$ is a divisible module. But any divisible module over a principal ideal domain is injective ([8], Chap. II, Theorem 3). Consequently $M$ is injective.

Let $M$ be a torsion module over a principal ideal domain $R$. Let $p$ be any irreducible element of $R$. $M$ is said to a $p$-module if for each $x \in M$, $x\,p^k=0$ for some positive integer $k$. $M$ is said to be a *cyclic module* in case $M=x\,R$ for some $x \in M$. $M$ is said to be a *quasi cyclic $p$-module* if $M$ is a $p$-module, containing a countable number of non-zero elements $x_1, x_2, \ldots, x_n, \ldots$ which generate $M$ and further $x_1\,p=0$, $x_{i+1}\,p=x_i$ for every $i$. (We know that any abelian group can be regarded as module over the ring $Z$ of integers and $Z$ is a principal ideal domain. The above concepts are simple generalizations of the concept of abelian $p$-group, cyclic groups, quasi cyclic $p$-groups oı type $p$, where $p$

is any prime integer. These concepts for abelian groups are discussed by FUCHS in [2] (Chap. I, Sect. 3, 4, 7)). A module $M$ is said to be a *weakly cyclic* $p$-module if it is either a cyclic $p$-module, or a quasi cyclic $p$-module. Any homomorphic image of a weakly cyclic $p$-module is a weakly cyclic $p$-module. Any two weakly cyclic $p$-modules are isomorphic if any only if they have same annihilator ideals. Any proper submodule of a weakly cyclic $p$-module is a cyclic $p$-module. The family of all submodules of a weakly cyclic $p$-module is totally ordered under the inclusion relation. Thus a weakly cyclic $p$-module does not have any proper direct summand. By usual application of ZORN's lemma it can be easily seen that any weakly cyclic $p$-submodule of a torsion module is contained in a maximal weakly cyclic $p$-submodules. (The proofs for the above observations are almost on the same line as for Abelian $p$-groups, $p$ any prime integer.) For any submodule $N$ of $M$ $\mathrm{ann}_R(N)$ denotes the ideal $\{a \in R\colon\ Na = (0)\}$. It can be easily seen that if $M$ is quasi-cyclic $p$-module then $\mathrm{ann}_R(M) = (0)$. This fact we denote by saying that $\mathrm{ann}_R(M) = p^\infty R$.

(3.2) Lemma. *Let $M$ be any pseudo-injective torsion module over a principal ideal domain. Let $p$ be any irreducible element of $R$. Then any two maximal weakly cyclic $p$-submodules of $M$ are isomorphic.*

Proof. Let $N_1$, $N_2$ be any two maximal weakly cyclic $p$-submodules. Let $\mathrm{ann}_R(N_1) = p^n R$, $\mathrm{ann}_R(N_2) = p^m R$, where $n$ and $m$ are either non negative integers or else any one of them may be infinity in case any one of $N_1$, $N_2$ is quasi cyclic. Now $N_1$ is isomorphic to $N_2$ if and only if $n = m$. Let us suppose $n > m$. In this case $m \neq 0$ since $m = 0$ implies $N_2 = (0) \subsetneqq N_1$. For $n$ following possibilities are there: (i) $n$ is infinite, (ii) $n$ is finite. Let $n$ be infinite. In this case $N_1$ is a quasi-cyclic $p$-submodule. Thus it has a countable set of nonzero members $x_1, x_2, \ldots, x_n, \ldots$ which generate $N_1$ and $x_1 p = 0$, $x_{i+1} p = x_i$ for every $i$. If we put $N_2' = x_m R$ then $\mathrm{ann}_R(N_2') = p^m R$. Consequently $N_2' \simeq N_2$. Let now $n$ be finite. In this case $N_1 = yR$ for some $y \in N$. Put $N_2' = y R p^{n-m}$. Then $N_2'$ is a submodule of $N_1$ such that $\mathrm{ann}_R(N_2') = p^m R$. Again $N_2' \simeq N_2$. In any case we find that $N_1$ contains a submodule $N_2'$ isomorphic to $N_2$. Let $\psi_2'\colon\ N_2' \to N_2$ be an isomorphism of $N_2'$ on to $N_2$. Since $M$ is pseudo-injective, we can find an $R$-endomorphism $\varphi$ of $M$ which is an extension of $\psi$. Then $\varphi(N_1)$ is a weakly cyclic $p$-submodule containing $N_2$. Consequently $\varphi(N_1) = N_2$ because of maximality of $N_2$. Then we have

$$N_1 = \ker\varphi \oplus N_2'.$$

Now $\ker\varphi \neq (0)$ as $N_2' \neq N_2$. Also $N_2' \neq (0)$. This contradicts the fact that a weakly cyclic $p$-module cannot have any direct summand. Hence $n \leqslant m$. Similarly $m \leqslant n$. Thus $n = m$ and $N_1 \simeq N_2$.

(3.3) Theorem. *Any pseudo-injective module over a principal ideal domain is quasi injective.*

Proof. Let $M$ be any pseudo-injective module over a principal ideal domain $R$. If $M$ is not a torsion module then by Lemma (3.1) $M$ is injective.

Let us now suppose that $M$ is a torsion module. Let $N$ be any submodule of $M$ and $\sigma$: $N \to M$ any $R$-homomorphism. By usual application of ZORN's lemma we can find a submodule $N'$ of $M$ containing $N$ and an $R$-homomorphism $\eta$: $N' \to M$ which is an extension of $\sigma$ such that $\eta$ has no further extension. We want to show that $N' = M$. Let $N' \neq M$. Then $M/N'$ is a non-zero torsion $R$-module. Then there exist a non-zero element $\bar{y} = y + N'$ of $M/N'$ such that $\operatorname{ann}_R(\bar{y}) = p\,R$ for some irreducible element $p$ of $R$. Let $\operatorname{ann}_R(y) = a\,R$ then $a \neq 0$, since $M$ is a torsion module. Then $\bar{y}\,a = \bar{0}$. This gives $p$ divides $a$. We can write $a = b\,p^{\alpha}$, where $b$ is such that highest common factor of $b$ and $p$ is equal to 1. Then $\overline{y\,b} = y\,b + N$ is such that $\operatorname{ann}_R(\overline{y\,b}) = p\,R$, $\operatorname{ann}_R(y b) = p^{\alpha} R$. Thus without loss of generality we may assume that $\bar{y}$ is a non-zero member of $M/N'$ such that $\operatorname{ann}_R(\bar{y}) = p\,R$, $\operatorname{ann}_R(y) = p^{\alpha}\,R$.

Now $y\,p \in N'$, as $\bar{y}\,p = \bar{0}$. Thus $\eta(y\,p)$ is defined. Either $\eta(y\,p) = 0$ or $\eta(y\,p) \neq 0$. Let $N_1$ be the submodule of $M$ generated by $N' \cup \{y\}$. Let us suppose $\eta(y\,p) = 0$. Define $\eta'$: $N_1 \to M$ such that $\eta'(x + y\,a) = \eta(x)$ for every $x \in N'$, $a \in R$. By using the fact that $\eta(y\,p) = 0$, we can show that $\eta'$ is well defined $R$-homomorphism. Further $\eta'$ is an extension of $\eta$. Since $N_1 \neq N'$, therefore $\eta'$ is a proper extension of $\eta$. This is a contradiction. Thus we have $\eta(y\,p) \neq 0$. As $y\,p^{\alpha} = 0$, therefore $\eta(y\,p)p^{\alpha-1} = 0$. Thus $\operatorname{ann}_R[\eta(y\,p)] = p^{\beta}\,R$ for some $\beta \leqslant \alpha - 1$. Now $y\,R$ and $\eta(y\,p)R$ are both non-zero cyclic $p$-submodules, they are contained in two maximal weakly cyclic $p$-submodules say $N_2$, $N_3$ respectively. By Lemma (3.2) $N_2 \simeq N_3$ and $\operatorname{ann}_R(N_2) = \operatorname{ann}_R(N_3) = p^{\gamma}\,R$ for some $\gamma \geqslant \alpha$. Then $N_3$ contains a cyclic $p$-submodule say $N_4$ isomorphic to $yR$. Then $\operatorname{ann}_R(N_3) = \operatorname{ann}_R(yR) = p^{\alpha}R \subsetneq p^{\beta}R = \operatorname{ann}_R(\eta(y\,p)R)$. Thus we have $N_4 \supsetneq \eta(yp)R$. Now it can be easily seen that there exist $z \in N_4$ such that $\eta(y\,p) = z\,p$. In this case again define $\xi$: $N_1 \to M$ such that $\xi(x + y\,a) = \eta(x) + z\,a$ for every $x \in N'$, $a \in R$; $\xi$ is a well defined $R$-homomorphism and it is a proper extension of $\eta$. We again get a contradiction. Thus we must have $N' = M$. Consequently $\eta$ is an $R$-endomorphism of $M$ which is an extension of $\sigma$. Hence $M$ is quasi-injective.

4. – Let $K$ be any right ORE domain with unity. Let the division ring $D$ be the classical right quotient ring of $K$, $n$ be any positive integer, $K_n$ and $D_n$ be $n \times n$ - matrix rings over $K$, and $D$ respectively. Then $D_n$ is a classical right quotient ring of $K_n$, and $K_n$ is a prime ring. Let $R$ be any ring such that $K_n \subseteq R \subseteq D_n$. Then $R$ is a prime right GOLDIE ring having $D_n$ as its classical

right quotient ring. Further $R$ and $D_n$ are both right quotient rings of $K_n$ in the sense defined by JOHNSON [3]. Let $M$ be any $R$-module.

(4.1) Lemma. *$M^{\Delta} = (0)$ if and only if $M$ is torsion free.*

Proof. It is proved by GOLDIE in [1] (Theorem (3.9)) that any right ideal of $R$ is essential if and only if it contains a regular element. Now any element $x \in M$ is a torsion element if and only if $\operatorname{ann}_R(x)$ contains a regular element. Thus $x$ is a torsion element if and only if $\operatorname{ann}_R(x)$ is an essential right ideal. This implies that $M^{\Delta} = (0)$ if and only if $M$ is a torsion free $R$-module.

Hence forth we shall assume that $M$ is a torsion free $R$-module. Since $K_n \subseteq R$ we can regard $M$ as a $K_n$-module. For any sub-ring $T$ of $R$, $M^{\Delta}(T)$ denotes the $T$-singular submodule of $M$. Let $\{e_{ij}$: $i, j = 1, 2, \ldots, n)$ be the matrix units of $K_n$ and $D_n$. Now $Me_{11}$ can be regarded as a $K$-module. By defining for any $x\,e_{11} \in M\,e_{11}$, $a \in K$, $(x\,e_{11})\,a = x\,(a\,e_{11})$.

(4.2) Lemma. *$M$ is a torsion free $K_n$-module.*

Proof. Since $R$ is a JOHNSON right quotient ring of $K_n$ by [5] (Lemma (2.2)), $M^{\Delta}(R) = M^{\Delta}(K_n)$. By Lemma (4.1) $M^{\Delta}(R) = (0)$. Consequently $M^{\Delta}(K_n) = 0$. Taking $R = K_n$ in Lemma (4.1) we conclude that $M$ is a torsion free $K_n$-module.

(4.3) Lemma. *$M\,e_{11}$ is a torsion free $K$-module.*

Proof. Let $x\,e_{11} \in M\,e_{11}$ and $a(\neq 0) \in K$ such that $(x\,e_{11})a = 0$. The element $\sum_{i=1}^{n} a\,e_{ii}$ is a regular element of $K_n$ and $(x\,e_{11}) \sum_{i=1}^{n} a\,e_{ii} = 0$. This gives $xe_{11} = 0$ as $M$ is a torsion free $K_n$-module. Hence $Me_{11}$ is also torsion free $K$-module.

Let $R'$ be any ring with unity, $R'_n$ be the $n \times n$ matrix ring over $R'$. Let $N$ be any $R'_n$-module. Then $Ne_{11}$ can be regarded as an $R'$-module. LEVY [6] (Corollary (2.3)) proved that $N$ is an injective $R'_n$-module if and only if $Ne_{11}$ is an injective $R'$-module. By similar arguments we can prove that $N$ is a pseudo-injective $R'_n$-module if and only if $Ne_{11}$ is a pseudo-injective $R'$-module. We state without proof the following

(4.4) Lemma. *If $M$ is a pseudo-injective $K_n$-module, then $M\,e_{11}$ is a pseudo injective $K$-module.*

(4.5) Lemma. *If $M$ is a pseudo-injective $R$-module, then $M$ is also a pseudo-injective $K_n$-module.*

Proof. Let $N$ be any $K_n$-submodule of $M$ and $\sigma$: $N \rightarrow M$ any $R$-monomorphism. Now by Lemma (4.2) $M$ is a torsion free $K_n$-module. Since $S = D_n$ is a classical rigt quotient ring of $K_n$, by [6] (Corollary (1.6)), $\sigma$ can be extended to a $S$-monomorphism $\sigma'$: $NS \rightarrow MS$. Let $N_1 = \sigma'^{-1} [\sigma'(NS) \cap M] \cap M$. Then $N_1$ is an $R$-submodule of $M$ containing $N$ such that $\sigma'(N_1) \subseteq M$. Let $\eta$ be the restriction of $\sigma'$ to $N_1$. Then $\eta$ is an $R$-monomorphism of $N_1$ into $M$, $\eta$ coincides with $\sigma$ on $N$. As $M$ is a pseudo-injective $R$-module, therefore $\eta$ can be extended to an $R$-endomorphism $\xi$ of $M$. Clearly $\xi$ is a $K_n$-endomorphism of $M$, and $\xi$ is an extension of $\sigma$. Hence $M$ is a pseudo-injective $K_n$-module.

(4.6) Lemma. *$\hat{M} = MS$, where $\hat{M}$ denotes injective hull of $M_R$ and $S = D_n$.*

Proof. Since $M$ is a torsion free $R$-module and $S$ is a classical right quotient ring of $R$, $M$ can be embedded in a $S$-module $MS$. Now $MS$ is a torsion free divisible $R$-module. Thus by a result of LEVY [6] (Theorem (3.3)), $MS$ is an injective $R$-module; and $\hat{M}_R \subset MS$. The fact that every element of $MS$ is of the form $m\, d^{-1}$, where $m \in M$, $d \in R$ and $d$ regular, implies that $MS$ is an essential extension of $M$ as an $R$-module. Consequently $MS \subset \hat{M}$, since $\hat{M}$ is the maximal essential extension of $M$ as $R$-module. Hence we get $MS = \hat{M}$.

(4.7) Theorem. *Any torsion free pseudo-injective $R$-module is injective.*

Proof. Let $M$ be any torsion free pseudo-injective $R$-module. Then the Lemmas (4.2) and (4.5) implies that $M$ is a torsion free pseudo-injective $K_n$-module. By using Lemmas (4.3) and (4.4), we get that $Me_{11}$ is a torsion free pseudo-injective $K$-module. Thus for any $x \in M e_{11}$, $a \in K$ and $x\, a = 0$ implies $x = 0$ or $a = 0$. Now we show that $Me_{11}$ as a divisible $K$-module. Let $x \in M\, e_{11}$, $a \in K$, $a \neq 0$, $N = x\, a\, K$. The mapping $\sigma$: $N \rightarrow M\, e_{11}$ such that $\sigma(x\, a\, b) = x\, b$ for every $b \in K$ is a $K$-monomorphism. It can be extended to a $K$-endomorphism $\eta$ of $Me_{11}$. Let $\eta(x) = y$. Then $y\, a = \eta(x\, a) = \sigma(x\, a) = x$. Hence $Me_{11}$ is a torsion free divisible $K$-module. As $K$ has $D$ as its classical right quotient ring, therefore by [6] (Theorem (3.3)), $M\, e_{11}$ is an injective $K$-module. Thus $M$ is an injective $K_n$-module. By [7] (Theorem (3.1)), $M$ is a divisible $K_n$-module. As $M$ is torsion free as well as divisible $K_n$-module, therefore $M$ can be made into an $S$-module. That gives $MS = M$. But by Lemma (4.6), $\hat{M} = MS$. Hence $M$ is an injective $R$-module.

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