Riv. Mat. Univ. Parma (2) 9 (1968), 247-260

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The Influence on a Finite Group of Certain Types of its Proper Subgroups. (**)

Introduction.

B. HUPPERT ([6], Satz 22) proved that if all the proper subgroups of the finite group G are supersolvable, then G is solvable. HUPPERT's result is an extension of the famous SCHMIDT-IWASAWA theorem. More recently J. S. ROSE ([10], p. 589) extended HUPPERT's theorem to the following: «If every proper self-normalizing subgroup of G has a SYLOW tower complexion $p_1, p_2, \dots, p_n$; then G is solvable.» R. BAER ([1], [2], [3]) introduced the concept of finite σ-dispersed groups, where σ is a partial order in a set of primes π. BAER's concept is more general than the SYLOW tower property of finite groups. In ([1], p. 172) BAER showed that if every proper subgroup of the finite group G is σ-dispersed, σ a partial order in the set of all primes, then G is solvable. Among the results of the present paper is that the finite group G is solvable if every proper self-normalizing subgroup of G is σ-dispersed, where σ is a partial order in the set of all primes. Actually a much more general result than the one mentioned above is obtained in the present paper. The reader is referred to Theorem 1.

C. S. SAH [11] termed the finite group G semi-nilpotent if the normalizer in G of each nonnormal p-subgroup P, p a prime, of G is nilpotent. Among the many interesting results in [11], SAH proved that a semi-nilpotent group is solvable. We exetend SAH's result to the following: « The finite group G is solvable if the normalizer of each nonnormal p-subgroup P, is σ-dispersed. » Here σ is a partial order in the set of all primes.

The present paper furnishes some general procedures to obtain sufficient conditions for a finite group to be solvable. Let Δ denote a class of finite groups

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(**) The author was supported by National Science Foundation Grant G. P. 5948. — Ricevuto: 2-VIII-1968.

such that Δ is closed under the operations of forming quotient-groups and subgroups and we assume that if every proper subgroup of the finite group G is a Δ-group, then G is solvable. We prove that if the abnormal maximal subgroups of each normal subgroup of the finite group G is a Δ-group, then G is solvable. A finite group G is called an almost Δ-group if the normalizer of each nonnormal p-subgroup P of G, p a prime, is a Δ-group. Almost Δ-groups are solvable.

In the final section of the present paper we consider almost σ-dispersed groups where again σ is a partial order in the set of primes. If p is a σ-maximal prime divisor of the order of the finite group G and the SYLOW p-subgroups of G are Abelian, then G contains a normal SYLOW p-subgroup or G is p-nilpotent.

The author wishes to thank Professor REINHOLD BAER for several stimulating conversations on the subjects contained in this paper. Professor BAER's general approach to solvability questions has been very helpful to author.

1. - Notations and definitions.

The only groups considered here are finite.

If H is a subset of a group G, then: $H^x = x^{-1}Hx$ for each $x \in G$, $\{H\}$ is the subgroup of G generated by H, $N_G(H)$ is the normalizer of H in G, $C_G(H)$ is the centralizer of H in G.

If H is a subgroup of a group G, then: H^1 is the first derived subgroup of H, $\varphi(H)$ is the FRATTINI subgroup of H, $|H|$ denotes the order of H, $H < G$ means H is a proper subgroup of G.

Let H be a subgroup of a group G. We define the hypernormalizer of H in G as follows:

Let $H_0 = H$, and for each positive integer i, $H_i = N_G(H_{i-1})$. We have $H_0 \leqslant H_1 \leqslant H_2 \leqslant \ldots$, and since G is finite, this ascending chain of subgroups terminates. The subgroup reached is the hypernormalizer of H (see [9]).

The subgroup H of the finite group G is called self-normalizing if H is its own normalizer in G. Thus H self-normalizing if and only if H coincides with its hypernormalizer.

The subgroup H of the finite group G is called an *abnormal subgroup* of, G if, for each $g \in G$, $g \in \{H, H^g\}$; or equivalently, as shown by CARTER in [4] and [5] if H satisfies the two conditions:

(a) every subgroup of G containing H is self-normalizing in G;

(b) H is not contained in two distinct conjugate subgroups of G.

We recall the obvious but convenient fact that a maximal subgroup of G is either self-normalizing or normal. Hence, the abnormal maximal subgroups of G are precisely its self-normalizing maximal subgroups. R. W. CARTER showed in [5] that a finite solvable group G possesses nilpotent self-normalizing subgroups, that all such subgroups are abnormal and conjugate in G. These subgroups of G are often called the CARTER subgroups of G.

A group G is called a *Sylow tower group* if every nontrivial homomorphic image of G has a nontrivial normal SYLOW subgroup (see [6], [10]). This is true if and only if for some ordering of distinct prime numbers $p_1, p_2, \ldots, p_n$, there exists a series of normal subgroups of G:

$$1 = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_n = G ,$$

such that the factor group G_i/G_{i-1} is isomorphic to a SYLOW p_i-subgroup of G ($i = 1, 2, \ldots, n$). Such a series will be called a SYLOW tower of G of complexion $p_1, p_2, \ldots, p_n$ (see [10]). G_i/G_{i-1} is permitted to be trivial, and this happens if and only if p_i does not divide $|G|$. In any case, if G has a SYLOW tower complexion $p_1, \ldots, p_n$, then all the prime factors of $|G|$ appear among $p_1, p_2, \ldots, p_n$. We note that if G has a SYLOW tower complexion $p_1, p_2, \ldots, p_n$, then subgroups and quotients of G have SYLOW towers of the same complexion (see [10]).

Let the group G have a SYLOW tower complexion $p_1, p_2, \ldots, p_n$. If $p_1 > p_2 > \ldots > p_n$, then G is said to have an ordered SYLOW tower. One of the most striking properties of supersolvable groups is that they always possess ordered SYLOW towers (see [6], Satz 7).

Let π denote a set of prime numbers. Then P_π will denote the set of primes not in π. An element x in the group G is called a *π-element* if the order of x is divisible by primes in π only. The group G is called a *π-group* if $|G|$ is divisible by primes in π only. The group G is termed *π-separated* if its composition factors are either π-groups or P_π-groups (see [2]). Thus π-separation and P_π-separation are equivalent properties. We note that a solvable group is π-separated for every set of primes π. Further, subgroups and quotient groups of π-separated groups are π-separated (see (2]). The group G is called *π-homogeneous* if $N_G(S)/C_G(S)$ is a π-group for each π-subgroup S of G (see [2]).

Once again let π denote a set of primes. The group G is called *π-closed* if the products of π-elements in G are π-elements. We note that π-closure is equivalent to the requirement that the set G_π of π-elements of G is a characteristic subgroup of G (see [1], [2]). If set π consists of one prime p only, then we speak of p-closure which amounts to requiring the existence of a normal SYLOW p-subgroup and also p-nilpotency amounts to P_p-closure. We note that subgroups and factor groups of π-closed groups are π-closed. Many interesting properties of π-closed groups can be found in [1], [2].

Next we consider a partial ordering σ in the nonempty set of primes π. We shall write $p \, \sigma \, q$ whenever p and q are distinct in π and p precedes q in the partial order σ. If we have equality of primes, then we always write $p \overset{\sigma}{=} p$ whenever p is a prime in π. Then we note that $p \sigma p$ is false for every prime p in π. Further, $a\sigma b$ and $b\sigma c$ implies $a\sigma c$, where a, b and c are primes in π (see [1],[2], [3]). A *σ-segment* of π is a subset Γ of π with the following property: if p belongs to Γ and $q\sigma p$, then q too belongs to Γ (see [1], [2], [3]). The finite group G is called *σ-dispersed* if G is Γ-closed for every σ-segment Γ of π (see [1], [2], [3]). We mention that subgroups and factor groups of σ-dispersed groups are σ-dispersed (see [1], [2], [3]). The product of normal σ-dispersed subgroups of the finite group G is a normal σ-dispersed subgroup of G (see [1]).

If $\pi(G)$ denotes the collection of prime divisors of $|G|$ belonging to π, then σ defines a partial ordering on $\pi(G)$; and G is clearly σ-dispersed for the partial ordering σ of π if and only if G is σ-dispersed for the partial ordering σ of $\pi(G)$. The set $\pi(G)$ is called the set of *relevant primes* of π (see [1]). An element p of $\pi(G)$ is called σ-minimal if we have $q \not\sigma p$ for every prime q of $\pi(G)$. We note that σ-maximal primes are defined similarly. We also note that the group G is σ-dispersed if and only if $G/\varphi(G)$ is σ-dispersed (see [1], p. 165). Another intersting characterization of σ-dispersion is the following: the group G is σ-dispersed if and only if every subgroup S of G is p-closed for every σ-minimal prime p in $\pi(S)$ (see [2], Thm. 1.1).

Now let π be a nonempty set of primes and let σ be a partial order in π. If π is the set of all primes, and if σ is the trivial partial order on π (i. e. $p \not\sigma q$ for every pair of primes), then σ-dispersion and nilpotency are equivalent concepts (see [1], p. 173). If π is the set of all primes and σ some complete ordering on π, then the group G being σ-dispersed amounts to G being a Sylow tower group relative to some complexion of primes (see [2], Thm. 1.1).

Let π be the set of all primes and let η denote the natural ordering in π. We recall that $p \eta p$ is false for each p in π, and that we write $p \overset{\eta}{=} p$ for each p in π. Let σ denote the inverse ordering of η and then σ is a partial ordering in π. We also note that $p \sigma p$ is false for each p in π, but $a \sigma b$ and $b \sigma c$ imply $a \sigma c$, where a, b and c are primes. Then the group G being σ-dispersed amounts to G having an ordered Sylow tower.

2. - Some extensions of the Schmidt-Iwasawa theorem.

A group theoretical property θ defines a class of finite groups. Thus every finite group either has property θ or does not have this property. It will be convenient to term θ-group every finite group with property θ. Throughout we shall assume of such a property θ that the *identity group* is a θ-group.

For the remainder of this paper we assume that θ is a group theoretic property such that:

θ_1. *Subgroups of θ-groups are θ-groups.*

θ_2. *Homomorphic images of θ-groups are θ-groups.*

θ_3. *If H is a normal subgroup of the finite group G such that H and G/H are θ-groups, then G is a θ-group.*

θ_4. *The class of finite Abelian groups is a subclass of θ.*

We remark that θ_1 is known as the subgroup-inherited property of θ and θ_2 as the homomorphism-invariant property of θ (see [**1**]). Let $\sum$ denote a (non-empty) set of primes. Then the class of finite $\sum$-separated groups satisfies θ_1 through θ_4. In particular, the class of finite solvable groups satisfies θ_1 through θ_4.

Let Δ be a group theoretic property such that:

Δ_1. *Subgroups of Δ-groups are Δ-groups.*

Δ_2. *Homomorphic images of Δ-groups are Δ-groups.*

Δ_3. *If every proper subgroup of the finite group G is a Δ-group, then G is a θ-group.*

Remark 1. The class of finite Δ-groups is a subclass of the class of groups θ.

In the present article we assume that certain proper subgroups of a finite group G are Δ-groups and then prove that G is a θ-group. In the results to follow the reader will no doubt notice the importance of Δ_3. The general procedure used here was suggested by REINHOLD BAER, however the basic motivation is the now famous SCHMIDT-IWASAWA theorem.

Now let θ denote the class of finite solvable groups. The SCHMIDT-IWASAWA theorem is the following:

(I) *If a finite group G has all its proper subgroups nilpotent, then G is solvable.*

The hypothesis in (I) has in fact much stronger implications for the structure of G than solvability (see [**7**], [**8**], [**12**]). J. S. ROSE [**9**] generalized (I) to the following:

(II) *If a finite group G has all its proper abnormal subgroups nilpotent, then G is solvable.*

B. HUPPERT ([**6**], Satz 22) showed that the statement (I) may be improved to the following:

(III) *If a finite group G has all its proper subgroups supersolvable, then G is solvable.*

We remark that the hypothesis in (III) has much stronger implications for the structure of G than solvability. For example if the order of G has at least four distinct primes, then G is supersolvable (see [6], Satz 22). J. S. ROSE [9] generalized (III) to the following:

(IV) *If a finite group has all of its proper self-normalizing subgroups supersolvable, then G is solvable.*

Remark 2. In the hypothesis of (IV) one can not replace the phrase « proper self-normalizing subgroups » by « proper abnormal subgroups ». The reader is referred to Example 1.

We recall that a supersolvable group always has an ordered SYLOW tower (see [6], Satz 7). Hence, the next result of ROSE ([10], Thm. 8) generalizes (III). Moreover, (VI), which is also a result of ROSE ([10], p. 589), is a generalization of (IV) and (V).

(V) *If every proper subgroup of the finite group G has a Sylow tower complexion $p_1, p_2, \ldots, p_n$, then G is solvable.*

(VI) *If every proper self-normalizing subgroup of the finite group G has a Sylow tower complexion $p_1, p_2, \ldots, p_n$, then G is solvable.*

We now give a result of R. BAER ([1], p. 172) which is more general than (I), (III), or (V).

(VII) *Let σ denote a partial ordering in the set $\sum$ of all primes. If every proper subgroup of a finite group G is σ-dispersed, then G is solvable.*

From the results stated in (I) through (VII) we note that for the class θ of finite solvable groups we can take Δ to be the class of finite σ-dispersed groups, σ a partial order in the set of all primes. Of course, we can take Δ to be the various classes of finite groups made up of SYLOW tower groups of the same complexion, supersolvable groups, or nilpotent groups.

Theorem 1. *Let G be a finite group such that the abnormal maximal subgroups of each normal subgroup of G are Δ-groups. Then G is a θ-group.*

Proof. Suppose that the theorem were false. Then there would exist a finite group G with the following properties:

(1) G is not a θ-group.

(2) If H is a normal subgroup of G and K is an abnormal maximal subgroup of H, then K is a Δ-group.

(3) If L is a finite group whose order is smaller than the order of G and L satisfies (2), then L is a θ-group.

The group G is not simple because of Δ_3 and (1). Let $N \neq 1$ be a normal subgroup of G and let S/N be an abnormal maximal subgroup of M/N, where

M/N is a (nontrivial) normal subgroup of G/N. Then S is an abnormal maximal subgroup of the normal subgroup M of G. Hence, S is a Δ-group so that S/N is a Δ-group because of Δ_2. Because of (3) it follows that G/N is a θ-group. From θ_3 we have the following:

(4) G contains a minimal normal subgroup M such that M is not a θ-group but G/M is a θ-group.

Let L be a maximal subgroup of M. If L is a normal subgroup of M, then $M^1 \leqslant L < M$ so that $M^1 = 1$ because of (4). By θ_1, M is a θ-group which contradicts (4). Hence, each maximal subgroup of M is abnormal in M. Because of Δ_2 and Δ_3 M is a θ-group and this fact again contradicts (4). This completes the proof.

From Theorem 1 and (VII) we obtain the following corollary which is a generalization of BAER's result (VII).

Corollary 1.1. *Let σ be a partial order in the set of all primes and let G be a finite group such that the abnormal maximal subgroups of each normal subgroup of G are σ-dispersed. Then G is solvable.*

From Theorem 1 and (III) we obtain the next corollary.

Corollary 1.2. *Let G be a finite group such that the abnormal maximal subgroups of each normal subgroup of G are supersolvable. Then G is solvable.*

The hypothesis in Corollary 1.2 can not be improved to the following: « If every proper abnormal maximal subgroup of the finite group G is supersolvable, then G is solvable. »

Example 1 (see [**9**], p. 351). Let $H = GL(3, 2)$, the general linear group of 3 by 3 matrices over the field of two elements. Then H is a simple group of order 168. Let f be the automorphism of H given by $f\colon\ x \to (x^{-1})^T$, where y^T is the transpose of the matrix y of H and y^{-1} is the inverse of y in H. Let G denote the relative holomorph of H by $\{f\}$, $\{f\}$ is a cyclic group of order two. Then G is not solvable and splits over H.

The abnormal maximal subgroups of G are supersolvable (see [**9**]). However, there exist abnormal maximal subgroups of H which are not supersolvable. For example, H contains an abnormal maximal subgroup which is isomorphic to the symmetric group on four symbols.

Lemma 1. *Let H be a subgroup of the finite group G. If S is a proper self-normalizing subgroup of H, then S is contained in a proper self-normalizing subgroup of G.*

Proof. Let N denote the hypernormalizer of S in G. If $N = G$, then S is subnormal in G so that S is a subnormal in H. This is impossible since S

is a proper self-normalizing subgroup of H, hence N is a proper subgroup of G. Since N is a self-normalizing subgroup of G, the lemma follows.

Remark 3. The substance of the argument in the above lemma was used by ROSE in proving the corollary to Theorem 3 of [**9**].

Let G be a finite group all of whose proper self-normalizing subgroups are Δ-groups. Let H be a normal subgroup of G and let S be an abnormal maximal subgroup of H. Then S is a proper self-normalizing subgroup of H so that S is a Δ-group by Lemma 1 and Δ_1. From Theorem 1 we obtain the following theorem.

Theorem 2. *Let G be a finite group all of whose proper self-normalizing subgroups are Δ-groups. Then G is a θ-group.*

Remark 4. We note that Theorem 2 is a generalization of ROSE's result (VI).

Remark 5. J. S. ROSE (see [**9**], p. 356) stated a general result which is contained in Theorem 2. ROSE's statement is the case in Theorem 2 when θ is the class of finite solvable groups.

3. - Almost Δ-groups.

In the remaining sections of this paper we take θ to be the class of all finite solvable groups.

A finite group G is called an almost Δ-group if for each nonnormal p-subgroup P, p a prime, $N_G(P)$ is a Δ-group.

Remark 6. Let Δ denote the class of finite nilpotent groups. SAH [**11**] termed an almost nilpotent group semi-nilpotent, and he determined many interesting properties of almost nilpotent groups. Among the properties that SAH established for almost nilpotent groups is that such a group is solvable. In this section we generalize some of SAH's results.

Theorem 3. *Let G be a finite almost Δ-group. Then G is solvable.*

Proof. We prove the theorem by using induction on $|G|$. Let K be a subgroup of G and let P be a nonnormal p-subgroup of K, p a prime. Then P is a nonnormal p-subgroup of G, hence $N_K(P) = N_G(P) \cap K$ is a Δ-group by Δ_1. By induction every proper subgroup of G is solvable.

We can assume that G is nonabelian. Assume by way of contradiction that G is simple. Let H be a maximal subgroup of G. Since H is solvable, it contains a nontrivial normal q-subgroup Q, q a prime. Then $N_G(Q) = H$ since G is simple. Therefore, every maximal subgroup of G is a Δ-group. By Δ_1 every proper subgroup of G is aΔ-group so that G is solvable by Δ_3. Thus G is Abelian which contradicts our assumption that G is nonabelian. Hence, we can take G to be a nonsimple group.

Let M be a minimal normal subgroup of G. Then M is a proper subgroup of G so that M is solvable. By Lemma 1 of ([**1**], p. 118), M is an elementary abelian q-group, q a prime. We now show G/M satisfies the conditions of the theorem. Let K/M denote a nonnormal r-subgroup of G/M, r a prime. We distinguish two cases.

Case 1. $r = q$. Then K is a nonnormal q-subgroup of G so that $N_G(K)$ is a Δ-group. Since $N_{G/M}(K/M) = N_G(K)/M$, $N_{G/M}(K/M)$ is a Δ-group by Δ_2.

Case 2. $r \neq q$. Then there exists a nonnormal r-subgroup L of G such that $LM = K$. Further, we note that $N_G(L)$ is a Δ-group since G is an almost Δ-group. Since

$$N_{G/M}(K/M) = N_{G/M}(LM/M) = N_G(L)M/M \cong N_G(L)/M \cap N_G(L),$$

it follows that $N_{G/M}(K/M)$ is a Δ-group by Δ_2.

In either case, $N_{G/M}(K/M)$ is a Δ-group so that G/M is an almost Δ-group. By our induction assumption G/M is solvable. Hence G is solvable since G/M and M are both solvable. This completes the proof.

By the proof of Theorem 3 and mathematical induction we obtain the following corollary.

Corollary 3.1. *Let G be a finite almost Δ-group. Then:*

(a) *If H is a subgroup of G, then H is an almost Δ-group.*

(b) *If H is a normal subgroup of G, then G/H is an almost Δ-group.*

From the remarks made in the previous section we obtain the following corollary to Theorem 3.

Corollary 3.2. *Let σ denote a partial ordering in the set of primes $\sum$. If the finite group G is almost σ-dispersed, then G is solvable.*

Remark 7. Corollary 3.2 is a generalization of (VII).

Next we present some examples of almost Δ-groups.

Example 2. Let σ denote the inverse of the natural ordering in the set of all primes $\sum$. We take Δ to be the class of finite σ-dispersed groups. Let S_4 (resp. A_4) denote the symmetric (resp. alternating) group on four symbols. Then $\sum(S_4) = \sum(A_4) = \{2, 3\}$ and $p = 3$ is the unique σ-minimal prime of both $\sum(S_4)$ and $\sum(A_4)$. Because of Theorem 1.1 of [**2**], S_4 and A_4 are not σ-dispersed groups, however S_4 and A_4 are almost σ-dispersed groups. Further, we note that $G = S_4 \times A_4$ is not an almost σ-dispersed group.

We conclude this section with some results whose content is similar to Theorem 4 of [11].

Lemma 2. *Let G be a finite almost Δ-group and let $G = H \times K$, where H and K are nontrivial normal subgroups of G. If H contains a nonnormal p-subgroup P, p a prime, then K is a Δ-group.*

Proof. Let P be a nonnormal p-subgroup of H. Then $N_G(P)$ is a proper subgroup of G, hence $N_G(P)$ is a Δ-group which contains K. By Δ_1, K is a Δ-group.

Throughout the remainder of this section σ will denote a partial order in the set of all primes $\sum$.

Theorem 4. *Let G be a finite almost Δ-group and let $G = H \times K$, where H and K are nontrivial normal subgroups of G. Then:*

(a) *If H is not σ-dispersed, then K is a Δ-group.*

(b) *If H and K are not σ-dispersed, then G is a direct product of Δ-groups.*

Proof. (a) Assume that H is not a σ-dispersed group. Because of Theorem 1.1 of [2], there exists a subgroup S of H and a σ-minimal prime p in $\sum(S)$ such that the Sylow p-subgroup P of S is nonnormal in S. Hence, P is nonnormal in H so that K is a Δ-group by Lemma 2.

(b) This is an easy consequence of (a).

We now assume that the class of Δ-groups satisfies the following additional property:

Δ_4. *The direct product of Δ-groups is a Δ-group.*

Remark 8. The class of finite σ-dispersed groups satisfies Δ_1 through Δ_4.

Corollary 4.1. *Let G be a finite almost Δ-group and let $G = H \times K$, where H and K are nontrivial normal subgroups of G. If H and K are not σ-dispersed, then G is a Δ-group.*

Proof. This follows from Theorem 4 (b) and Δ_4.

Corollary 4.2. *Let G be a finite almost σ-dispersed group and let $G = H \times K$, where H and K are nontrivial normal subgroups of G. Then either H or K is a σ-dispersed group.*

Proof. This is an immediate consequence of Corollary 4.1.

In the theorem to follow we use the fact that finite nilpotent groups are σ-dispersed. This fact is included in Theorem 1.1 of [2].

Theorem 5. *Let the finite group G be a direct product of the nontrivial normal subgroups H and K. Then:*

(a) *If G is almost σ-dispersed and H is not σ-dispersed, then K is either Hamiltonian or Abelian.*

(b) *If $(|H|, |K|) = 1$, H is Hamiltonian and K is an almost σ-dispersed group, then G is an almost σ-dispersed group.*

Proof. (a) Let G be an almost σ-dispersed group and assume that H is not σ-dispersed. Let P be a p-subgroup of K, p a prime. If $N_G(P) < G$, then H is a σ-dispersed group since $H \leqslant N_G(P)$. Hence, each primary subgroup of K is normal in K so that each subgroup of K being nilpotent is normal in K. This shows K is either Hamiltonian or Abelian.

(b) Let P be a p-subgroup of G, p a prime. Since the orders of H and K are relatively prime it follows that $P \leqslant H$ or $P \leqslant K$. If $P \leqslant H$, then P is normal in G since H is a Hamiltonian group. Hence, we assume that $P \leqslant K$. Then $N_G(P) = H \times N_K(P)$. If $N_K(P) < K$, then $N_G(P)$ is a σ-dispersed group since K is almost σ-dispersed. If $N_K(P) = K$, then P is normal in G. Therefore, we have shown that G is an almost σ-dispersed group.

4. - Almost σ-dispersed groups.

In the present section we devote our study to almost σ-dispersed groups, where σ is a partial order in the set $\sum$ of all primes. We show that a finite A-group which is almost σ-dispersed is a Sylow tower group. We begin with the following theorem.

Theorem 6. *Let G be a finite almost σ-dispersed group and let p be a σ-maximal prime in $\sum(G)$. Then either G contains a normal p-subgroup or G is Pp-closed.*

Proof. Assume that G is not Pp-closed. Because of Theorem 5.1 of [2] G is not p-homogeneous. Hence, G contains a p-subgroup Q such that $N_G(Q)/C_G(Q)$ is not a p-group. Therefore, there exists a Pp-element x of $N_G(Q)$ which is not contained in $C_G(Q)$. Assume by way of contradiction that $N_G(Q)$ is a proper subgroup of G. Since p is a σ-maximal prime of $\sum(N_G(Q))$ and $N_G(Q)$ is σ-dispersed, it follows by Theorem 1.2 of [2] that $N_G(Q)$ is Pp-closed. Let H denote the set of Pp-elements of $N_G(Q)$. Then H is a normal subgroup of $N_G(Q)$ which contains x and further $HQ = H \times Q$. This shows that x is an element of $C_G(Q)$ which is a contradiction. Hence, Q is a normal subgroup of G. This completes the proof.

Remark 9. There exist finite almost σ-dispersed groups G and a σ-maximal prime p in $\sum(G)$ such that G is not Pp-closed and G contains a normal p-subgroup which is not a SYLOW p-subgroup of G. For let σ denote the inverse of the natural ordering in $\sum$ and let S_4 denote the symmetric group on four symbols. Then S_4 is an almost σ-dispersed group and 2 is the unique σ-maximal prime of $\sum(S_4) = \{2, 3\}$ (see Example 2). We note that S_4 is not 3-closed and S_4 contains a normal 2-subgroup which is not a SYLOW 2-subgroup of S_4.

In view of Corollary 3.1, Theorem 6 has the following

Corollary 6.1. *Let G be a finite almost σ-dispersed group and let p be a σ-maximal prime in $\sum(G)$. Then G contains a normal p-subgroup P_0 (possibly trivial) such that G/P_0 is Pp-closed.*

Theorem 7. *Let G be a finite almost σ-dispersed group and let p denote a σ-maximal prime in $\sum(G)$. If the Sylow p-subgroups of G are Abelian, then G is either Pp-closed or p-closed.*

Proof. Let P be a SYLOW p-subgroup of G and assume that $N_G(P)$ is a proper subgroup of G; that is G is not p-closed. We note that p is a σ-maximal prime of $\sum(N_G(P))$. Because of Theorem 1.2 of [2], $N_G(P)$ is Pp-closed so that $N_G(P) = P \times R$, where R is the normal subgroup of $N_G(P)$ of all Pp-elements of $N_G(P)$. Since P is abelian, $C_G(P) = N_G(P)$ and therefore G is Pp-closed by BURNSIDE's theorem (see [13], Thm. 6.2.9). This completes the proof.

We recall that the finite solvable group G is termed an A-group if all the SYLOW subgroups of G are Abelian. Many interesting properties of A-groups can be found in TAUNT [14]. We note that a finite σ-dispersed group is an A-group whenever its SYLOW subgroups are Abelian. This fact is a consequence of Corollary 3.2.

Theorem 8. *Let G be a finite almost σ-dispersed group. If G is an A-group, then G contains a normal Sylow subgroup.*

Proof. We establish the theorem by induction on $|G|$. Let p be a σ-maximal prime of $\sum(G)$. By Theorem 7, G is either Pp-closed or G contains a normal SYLOW p-subgroup. If G contains a normal SYLOW p-subgroup, then the theorem follows. Hence, assume that G is Pp-closed. Then the set K of Pp-elements of G is a normal HALL subgroup of G. Because of Corollary 3.1, K is an almost σ-dispersed, and since K is an A-group, it follows by induction that K contains a normal SYLOW subgroup Q. Since H is a normal HALL subgroup of G, Q is a normal SYLOW subgroup of G. This completes the proof.

Remark 10. The assumption in Theorem 8 that G is an A-group can not be omitted. For let σ denote the inverse of the natural ordering in $\sum$ and let S_4 denote the symmetric group on four symbols. Then S_4 is an almost σ-dispersed group which is not an A-group (see Example 2). But S_4 does not contain a normal SYLOW subgroup.

Let $H \neq 1$ be a homomorphic image of the finite group G. Then H is an A-group if G is an A-group. Further, if G is almost σ-dispersed, then H is almost σ-dispersed by Corollary 3.1. Because of Theorem 8 we obtain the following theorem.

Theorem 9. *Let G be a finite almost σ-dispersed group. If G is an A-group, then G is a Sylow tower group.*

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Abstract.

In the present Note we determine several sufficient conditions for a finite group G to be solvable. One such condition is that the normalizes $N_G(P)$ in G of each nonnormal p-subgroup P of G, p a prime number, is supersolvable.

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