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On Some Fixed Point Theorems. (**)

1.1. – A mapping T of a metric space X into itself is said to satisfy LIPSCHITZ condition with LIPSCHITZ constant α if

$$d\big(T(p),\ T(q)\big) \leqslant \alpha\, d(p,\ q) \qquad (p,\ q \in X).$$

If this condition is satisfied with a LIPSCHITZ constant α such that $0 \leqslant \alpha < 1$, then T is called a contraction mapping. A well known theorem of BANACH states that:

If X is a complete metric space and T is a contraction mapping of X into itself, then there exists a unique point $\xi \in X$ such that $T(\xi) = \xi$.

1.2. – A mapping T: $X \to X$ of a metric space X into itself is said to be non expansive (ε-non expansive) if the condition

$$d\big(T(p),\ T(q)\big) \leqslant d(p,\ q)$$

holds for all $p,\ q \in X,\ p \neq q$ (for all $p,\ q$ with $d(p,\ q) < \varepsilon$). If we have strict inequality sign for all $p,\ q \in X,\ p \neq q$ (for all $p,\ q \in X$ such that $0 < d(p,\ q) < \varepsilon$). Then T is said « to be contractive (or ε-contractive) ».

Remark. The assumption $d\big(T(x),\ T(y)\big) < d(x,\ y)$ is not sufficient for the existence of a fixed point even on a complete metric space.

For example, let X be the set of real numbers with the usual metric. Define $T(x) = x + \pi/2 - \arctan x$.

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Since $\arctan x < \pi/2$ for every x, the operator T has no fixed point. At the same time if $x < y$, then

$$T(y) - T(x) = y - x - (\arctan y - \arctan x)\,,$$

and by LANGRANGE's formula

$$T(y) - T(x) = y - x - \frac{y - x}{1 + z^2} \qquad (x < z < y)\,.$$

If we had

$$|\,T(y) - T(x)\,| \geqslant |\,y - x\,|\,,$$

then this would mean that

$$\left|1 - \frac{1}{1 + z^2}\right| \geqslant 1\,,$$

but this inequality is not satisfied for any z and therefore we always have

$$|\,T(y) - T(x)\,| < |\,y - x\,|\,.$$

1.3. – A point $y \in Y \subset X$ is said to belong to the f-closure of Y, $y \in Y^f$, if $f(Y) \subset Y$ and there exists an $\eta \in Y$ and a sequence $\{n_i\}$ of positive integers $(n_1 < n_2 < \ldots < n_i < \ldots)$ so that $f^{n_i}(\eta) = Y$.

1.4. – A sequence $\{x_i\} \subset X$ is said to be an isometric (ε-isometric) sequence if the condition

$$d(x_m\,,\ x_n) = d(x_{m+k}\,,\ x_{n+k})$$

holds for all k, m, $n = 1, 2, \ldots$ with $d(x_m, x_n) < \varepsilon$. A point $x \in X$ is said to generate an isometric (ε-isometric) sequence under f if $\{f^n(x)\}$ is such a sequence.

2. – Theorem. *Let f and g be two commuting functions defined on a compact metric space X, then f and g have a common fixed point, provided that f satisfies the following properties:*

(i) $$d\big(f(x),\ f(y)\big) \leqslant d(x,\ y)\,,$$

(ii) $$\textit{if} \quad x \neq f(x) \quad \textit{then} \quad d\big(f(x),\ f^2(x)\big) < d\big(x,\ f(x)\big)\,.$$

Proof. The condition (i) implies that the sequence $\{d(f^n(x),\ f^{n+1}(x))\}$ is non increasing. Since the space is compact therefore the sequence will converge to a limit point, hence there exists a point y such that $y = \lim_k f^{n_k}(x)$. By (i) it is clear that f is continuous therefore $d(y, f(y)) = \lim_k d(f^{n_k}(x), f^{n_k+1}(x)) = \lim_n d(f^n(x),\ f^{n+1}(x)) = \lim_k d(f^{n_k+1}(x),\ f^{n_k+2}(x)) = d(f(y),\ f^2(y))$. Contradiction to condition (ii) unless $f(y) = y$. Thus y is a fixed point for f.

In order to show the uniqueness we assume that there exists another point $z \in X$, $z \neq y$ such that $f(z) = z$; then $d(f(y),\ f(z)) = d(y,\ z)$ contradicting (i), therefore $z = y$. Thus y is a unique fixed point of f. Now since g commutes with f therefore $fg(x) = gf(x)$ for all x. Again since $f(x) = x$, therefore $fg(x) = gf(x) = g(x)$, and $g(x)$ is also a fixed point for f. But f has a unique fixed point say x. Therefore $g(x) = x$, and thus x is a fixed point for g.

3.1. – In [3] Ward Cheney and Allen A. Goldstein have proved the following theorem:

« Let f be a map of a metric space X into itself such that:

(i) $d(f(x),\ f(y)) \leqslant d(x,\ y)$,
(ii) if $x \neq f(x)$, then $d(f(x),\ f^2(x)) < d(x,\ f(x))$,
(iii) for each x, the sequence $f^n(x)$ has a cluster point.

Then for each x the sequence $f^n(x)$ converges to a fixed point of f. »

Here we would like to remark that by relaxing conditions (ii) and (iii) we get a unique fixed point. Although the theorem has already been given by Michael Edelstein [6]. We prefer the direct rather simple proof here.

3.2. – Theorem. *Let f be a map of a compact metric space X into itself such that*

(i) $$d(f(x),\ f(y)) \leqslant dx(,\ y)$$

(equality sign occurs only when $x = y$). Then f has a unique fixed point.

Proof. The compactness of X and the condition (i) imply that each $x \in X^f$ generates an isometric sequence, Edelstein ([7], theorem 1′). Therefore by the definition of isometric sequence $d(x,\ f(x)) = d(f(x),\ f^2(x))$ but from the condition (i) we have $d(f(x),\ f^2(x)) \leqslant d(x,\ f(x))$.

This shows that $d(x, f(x)) = 0$ and which implies $x = f(x)$, i.e. x is a fixed point for f. To prove the uniqueness, let us assume that y is another point such that $y \neq x$ and $f(y) = y$. Then $d(f(x), f(y)) = d(x, y)$ contradicting condition (i) unless $x = y$. Thus x is a unique fixed point for f.

4. – Theorem. *Let T and K be two functions defined from a non-empty set X into itself such that K possesses a left inverse* (i.e. *a function K^{-1} such that $K^{-1}K = \mathrm{I}$, where* I *is the identity mapping of X*). *Then the function T has a fixed point if and only if KTK^{-1} has a fixed point.* (A similar result for right inverse has been given by Chu and Diaz [4].)

Proof. Suppose that x is a fixed point for T, then $Tx = x$ which implies $T(K^{-1}K)x = x$, or $KT(K^{-1}K)x = Kx$, or $(KTK^{-1})(Kx) = Kx$; i.e., Kx is a fixed point for KTK^{-1}.

Conversely, let us assume that y is a fixed point for KTK^{-1}. Then

$$KTK^{-1}y = y \quad \text{or} \quad K^{-1}KTK^{-1}y = K^{-1}y \quad \text{or} \quad TK^{-1}y = K^{-1}y,$$

i.e. $K^{-1}y$ is a fixed point for T.

5.1. – A mapping f of X into itself is said to be locally contractive if for every $x \in X$ there exists ε and λ $(\varepsilon < 0,\ 0 \leqslant \lambda < 1)$ which may depend on x such that $p, q \in S(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\}$ implies

$$d(f(p), f(q)) < \lambda\, d(p, q), \qquad p, q \in X, \quad p \neq q.$$

5.2. – A mapping f of X into itself is said to be (ε, λ) uniformly locally contractive if it is locally contractive and both ε and λ do not depend on x.

Remark. A globally contractive mapping can be regarded as (∞, λ) uniformly locally contractive mapping.

5.3. – A continuous mapping is eventually contractive if $0 < d(x, y)$ implies $\exists\, n(x, y) \in I^+$ (the positive integers) $\ni d(f^n(x), f^n(y)) < d(x, y))$, and it is ε-eventually contractive if f is continuous and

$$\exists\ \varepsilon > 0 \quad \text{such that} \quad 0 < d(x, y) < \varepsilon$$

implies

$$\exists\ n(x, y) \in I^+ \quad \text{such that} \quad d(f^n(x), f^n(y)) < d(x, y).$$

5.4. – A metric space X is said to be convex provided x and y in X implies there exists z in X such that $d(x, z) = d(z, y) = (1/2)\, d(x, y)$.

5.5. – A metric space X is said to be ε-chainable or well linked if for every pair (a, b) of points of X and for every $\varepsilon > 0$ there exists a ε-chain of finite sequence of points, of X, with $a = a_1, \dots, a_n = b$, such that $d(a_i, a_{i+1}) \leqslant \varepsilon$ for every $i < n$. In other words, a and b can be joined by a chain of steps at most equal to ε.

6. – Theorem. *Let X be a convex, compact ε-chainable metric space, and f a mapping of X into itself which is (ε, λ) uniformly locally contractive, then f is also eventually contractive.*

Proof. A theorem by Menger ([2], p. 41) states that a convex and complete metric space contains together with a, b also a metric segment whose extremeties are a and b, that is a subset isometric to an interval of lenght $d(a, b)$.

Using this fact we see that if $p, q \in X$ then there are points $p = x_1, x_2, \dots, x_n = q$ such that $d(p, q) = \sum_{i=1}^{n} d(x_{i-1}, x_i)$ and $d(x_{i-1}, x_i) < \varepsilon$. Hence

$$d(f(p), f(q)) \leqslant \sum_{i=1}^{n} d(f(x_{i-1}), f(x_i)) < \lambda \sum_{i=1}^{n} d(x_{i-1}, x_i) = \lambda\, d(p, q)\,.$$

By definition it is clear that $d(f(p), f(q)) < \lambda\, d(p, q)$ implies $0 < d(p, q) < \varepsilon \Longrightarrow d(f(p), f(q)) < d(p, q)$. Also, by definition every contractive mapping in the convex complete metric space may be regarded as ε-contractive mapping.

Now X is ε-chainable, therefore for distinct points p and q there exists $p = p_1, p_2, \dots, p_n = q$ such that $d(p_i, p_{i+1}) < \varepsilon$ for $i = 0, 1, \dots, n-1$. By Corollary 1 to theorem 2 [1], p_i is asymptotic to p_{i+1} under f for $i = 0, 1, \dots, n-1$. Hence there exists m in I^+ such that

$$d(f^m(p_i), f^m(p_{i+1})) < d(p, q)/n \qquad (i = 0, 1, \dots, n-1)\,.$$

Therefore

$$d(f^m(p), f^m(q)) \leqslant \sum_{i=0}^{n-1} d(f^m(p_i), f^m(p_{i+1})) < n\, d(p, q)/n = d(p, q).$$

7.1. – x is proximal to y under f provided for each $\alpha > 0$ there exists n a member of I^+ such that $d(f^n(x), f^n(y)) < \alpha$. If x and y are not proximal under

f they are said to be distal under f. If for each $\alpha > 0$ there exists n in I^+ such that $d(f^m(x), f^m(y)) < \alpha$ for all $m \geqslant n$, then x and y are said to be asymptotic under f. Note that we do not require $x \neq y$.

The following result has been proved by BAILEY [1].

Let X be a compact metric space and f be an ε-contractive mapping, i.e.

$$0 < d(x, y) < \varepsilon \quad \Longrightarrow \quad d(f(x), f(y)) < d(x, y) .$$

Then $d(x, y) < \varepsilon$ implies x and y are asymptotic under f.

7.2. – We prove the following

Theorem. *Let X be a compact ε-chainable metric space and f be ε-contractive mapping, i.e.*

$$0 < d(x, y) < \varepsilon \quad \Longrightarrow \quad d(f(x), f(y)) < d(x, y) .$$

Then every pair of points is aymptotic under f.

Proof. Since X is ε-chainable we define, for $p, q \in X$,

$$d(p, q) = \inf_{C(p,q)} \sum_{i=1}^{n} d(x_{i-1}, x_i) ,$$

where $C(p, q)$ denotes the collection of all ε-chains $p = x_0, x_1, \ldots, x_n = q$ [n arbitrary, $d(x_i, x_{i+1}) < \varepsilon$], holds. Indeed since f is ε-contractive we have

$$d(f(x_{i-1}), f(x_i)) < d(x_{i-1}, x_i) \qquad \text{provided} \quad d(x_{i-1}, x_i) < \varepsilon .$$

Hence

$$d(f(p), f(q)) < \inf_{C(p,q)} \sum_{i=1}^{n} d(f(x_{i-1}), f(x_i))$$

$$< \inf_{C(p,q)} \sum_{i=1}^{n} d(x_{i-1}, x_i) = d(p, q) ,$$

for all p, q. Thus the mapping is contractive.

Now since X is compact and f is contractive mapping of X into itself and therefore, by Theorem **3.2**, f contains a unique fixed point x, also the property compactness implies that each sequence $\{f^n(x)\}$ converges to x, therefore it follows that every pair of points is asymptotic under f.

8.1. – Following LUXEMBURG ([**9**], p. 541), the concept of a « generalized complete metric space » may be introduced in this quotation:

« Let X be an abstract set the elements of which are denoted by $x, y, \ldots$ and assume that on the cartesian product a distance function $d(x, y)$ $[0 < d(x, y) < \infty]$ is defined satisfaying the following conditions:

(D_1) $d(x, y) = 0$ if and only if $x = y$.

(D_2) $d(x, y) = d(y, x)$ (symmetry).

(D_3) $d(x, y) \leqslant d(x, z) + d(z, y)$ (triangle inequality).

(D_4) Every d-CAUCHY sequence in X is d-convergent, i.e. $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$ for a sequence $x_n \in X$ $(n = 1, 2, \ldots)$ implies the existence of an element $x \in X$ with $\lim\limits_{n\to\infty} d(x, x_n) = 0$ [x is unique by (D_1) and (D_3)]. »

This concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance. One might call such a space a generalized metric space.

8.2. – Denote by F the family of functions $\alpha(x, y)$ satisfying the following conditions:

(i) $\alpha(x, y) = \alpha\big(d(x, y)\big)$, i.e. α is dependent on the distance between x and y only.

(ii) $0 \leqslant \alpha(d) < 1$ for every $d > 0$.

(iii) $\alpha(d)$ is monotonically decreasing function of d.

In his paper A. F. MONNA [**10**] proved the following theorem:

« Let (X, d) be a complete generalized metric space. Let T_i $(i = 1, 2, \ldots)$ be a sequence of mappings of X into itself satisfying the following conditions:

There exists c and α $(c > 0,\ 0 < \alpha(d) < 1)$ so that $d(T_{ix}, T_{iy}) \leqslant \alpha(x, y)\, d(x, y)$ $(i = 1, 2, \ldots)$ whenever $d(x, y) \leqslant c$.

If $x_0 \in X$ then a positive integer $N(x)$ exists such that $n \leqslant N(x)$ implies

$$d(T_{n+k}(x_n),\ x_n) \leqslant c \qquad (k = 1, 2, \ldots).$$

Then the sequence of $\{x_n\}$, where $x_n = T_n x_{n-1}$ $(n = 1, 2, \ldots)$, converges and, if $y_0 = \lim\limits_{n\to\infty} x_n$, $\lim\limits_{k\to\infty} (T_k y_0) = y_0$. » [Here we have replaced α of given paper by $\alpha(x, y)$.]

9.1. – Here we prove that the assumption of MONNA's theorem imply even more stronger conclusion.

Theorem. *Let all assumptions of above theorem hold. Then a point* y *exists with the property that* $T_{n+k}(y) = y$ $(k = 1, 2, \dots)$.

In the proof of the above theorem we will use the following theorem and the proof of which is entirely analogous to that of the theorem 5.2. of [12] is omitted.

9.2. – If T is contraction mapping of a complete ε-chainable metric space X into itself satisfying

$$0 < d(x, y) < \varepsilon \quad \Longrightarrow \quad d(T(x), T(y)) \leqslant \alpha(x, y)\, d(x, y),$$

for every $x, y \in X$, and $\alpha(x, y) \in F$; then T has a unique fixed point.

Proof of Theorem in **9.1**. Consider an arbitrary fixed $x_0 \in X$; let $n \geqslant N(x_0)$ be fixed also. Let Y be the set of all $y \in X$ with the property that a sequence $C(y, x_n) \subset X$ exists, where $C(y, x_n) = \{y = p_0, p_1, p_2, \dots, p_l = x_n\}$ with $d(p_i, p_{i-1}) \leqslant C$ $(i = 1, 2, \dots, l)$. Obviously Y is closed metric subspace of X, also $T_{n+k}(Y) \subset Y$. Thus Y and T_{n+k} $(k = 1, 2, \dots)$ satisfy the assumptions of Theorem in 9.1. Therefore it follows that for each k, a unique ξ_k exists so that $T_{n+k}\, \xi_k = \xi_k$. To prove the theorem only we have to show that $\xi_1 = \xi_2 = \dots = \xi_k = (y)$.

Consider $T_{n+k}(T_{n+l}\, \xi_k) = T_{n+l}(T_{n+k}\, \xi_k) = T_{n+l}\, \xi_k$. Thus $T_{n+l}\, \xi_k$ is a fixed point under T_{n+k}. But T_{n+k} has a unique fixed point ξ_k. Therefore $T_{n+l}\, \xi_k = \xi_k$. Then ξ_k is a fixed point under T_{n+l}. Therefore by the uniqueness of ξ_k, $\xi_k = \xi_l$. Hence the theorem.

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