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Summability of Fourier Series by Karamata Method. (**)

1. – Let $f(t)$ be a periodic function with period 2π, and integrable in the sense of LEBESGUE over $(-\pi, \pi)$. Let its FOURIER series be given by

(1.1) $$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

We write

$$\varphi(t) = f(x + t) + f(x - t) - 2 f(x).$$

We define the numbers $\begin{bmatrix} n \\ \nu \end{bmatrix}$ by

(1.1′) $$x(x+1)(x+2)\dots(x+n-1) = \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix} x^{\nu},$$

where $n = 0, 1, 2, \dots;\ 0 \leqslant \nu \leqslant n$, and the numbers $\begin{bmatrix} n \\ \nu \end{bmatrix}$ are absolute values of STIRLING numbers of first kind.

Definition. The series $\sum a_n$, with the sequence of partial sum $\{S_\nu\}$, is said to be summable by KARAMATA method-K^λ, $\lambda > 0$, if the sequence

(1.2) $$S_n^\lambda = \left\{ \frac{\Gamma(\lambda)}{\Gamma(\lambda + n)} \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix} \lambda^\nu S_\nu \right\}$$

converges.

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The methods-K^λ were introduced by KARAMATA [4], who has shown that the methods are regular for $\lambda > 0$. AGNEW [1] applied these methods to FOURIER series and pointed out that even K^1 method is not FOURIER effective for continuous functions. For the first time in 1965, VLADATA-VUČKOVIĆ [7] has proved a positive result concerning the FOURIER effectiveness of these methods by proving:

Theorem. If

$$\varphi(t) = o(1/\log(1/t)), \tag{1.3}$$

as $t \to +0$, then the FOURIER series (1.1) is K^λ-summable at the point x to the sum $f(x)$, for every $\lambda > 0$.

HARDY [2] and IYENGAR [3] have proved that the condition (1.3) implies BOREL and harmonic summabilities, respectively. So the preceding theorem shows that the K^λ methods behave essentially as BOREL and harmonic methods regarding their FOURIER effectiveness. SAHNEY [5] and SIDDIQI [6] have generalised the results of HARDY [2] and IYENGAR [3], respectively. An analogous generalisation for K^λ-method is therefore expected. With this point of view we prove the following

Theorem. *If*

$$\Phi(t) = \int_0^t |\varphi(u)|\, du = o(t/\log(1/t)), \tag{1.4}$$

as $t \to +0$, *then the Fourier series* (1.1) *is* K^λ*-summable at the point* x *to the sum* $f(x)$, *for every* $\lambda > 0$.

2. – We need the following lemma to prove our theorem.

Lemma [7]. For $\lambda > 0$ and $0 < t < \pi/2$, we have

$$\frac{|\operatorname{Im} \Gamma(\lambda e^{it} + n)|}{\Gamma(\lambda \cos t + n) \cdot \sin(t/2)} = \frac{|\sin(\lambda \sin t \log n)|}{\sin(t/2)} + O(1)$$

uniformly in t, where Im denotes the imaginary part.

3. - Proof of the Theorem.

Let $S_\nu(x)$ denote the ν-th partial sum of the FOURIER series (1.1). We have

$$S_\nu(x) - f(x) = \frac{1}{2\pi}\int_0^\pi \varphi(t)\,\frac{\sin(\nu+(1/2))t}{\sin(t/2)}\,\mathrm{d}t + o(1)\,.$$

Let $S_n^\lambda(x)$ denote the transform (1.2) of $\{S_\nu(x)\}$. Then following VLADATA-VUČKOVIĆ [7], we have

$$S_n^\lambda(x) - f(x) = \{\Gamma(\lambda)/(2\pi)\}\int_0^\pi \varphi(t)\,K_n(t)\,\mathrm{d}t\,,$$

where

$$K_n(t) = \left\{\sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix} \lambda^\nu \sin(\nu+(1/2))t\right\} \Big/ \{\Gamma(\lambda+n)\sin(t/2)\}.$$

By (1.1′)

$$K_n(t) = \frac{\mathrm{Im}\,\{e^{it/2}\,\Gamma(\lambda e^{it}+n)/\Gamma(\lambda e^{it})\}}{\Gamma(\lambda+n)\sin(t/2)}\,,$$

where Im denotes the imaginary part.

Let δ be a positive number such that

$$1-\cos t > (1/3)\,t^2 \qquad \text{for } 0<t<\delta,$$

and A denote a constant independent of n and t and not necessarily the same at each occurance.

For $\delta<t<\pi$, $\varphi(t)$ is bounded and

$$|K_n(t)| \leqslant A\, n^{-\lambda(1-\cos\delta)}/\sin(\delta/2)\,.$$

Hence

$$\left|\frac{\Gamma(\lambda)}{2\pi}\int_\delta^\pi \varphi(t)\,K_n(t)\,\mathrm{d}t\right| \leqslant A\,\frac{n^{-\lambda(1-\cos\delta)}}{\sin(\delta/2)} = o(1)\,, \qquad \text{as } n\to\infty\,.$$

Therefore

$$| S_n^\lambda(x) - f(x) | \leqslant A \int_0^\delta | \varphi(t)\, K_n(t) | \, dt + o(1).$$

As

$$\frac{| \operatorname{Im} \{ e^{it/2}\, \Gamma(\lambda\, e^{it} + n)/\Gamma(\lambda\, e^{it}) \} |}{\sin(t/2)} \leqslant \frac{A \,|\, \operatorname{Im} \Gamma(\lambda\, e^{it} + n) \,|}{\sin(t/2)} + A \,|\, \operatorname{Re} \Gamma(\lambda\, e^{it} + n) \,|,$$

where Re means the real part, we obtain

$$| K_n(t) | \leqslant \frac{A \{ \Gamma(\lambda \cos t + n)/\Gamma(\lambda + n) \} \,|\, \operatorname{Im} \Gamma(\lambda e^{it} + n) \,|}{\Gamma(\lambda \cos t + n) \sin(t/2)} + A \,\frac{\Gamma(\lambda \cos t + n)}{\Gamma(\lambda + n)},$$

and, for $0 < t < \delta$,

$$\frac{\Gamma(\lambda \cos t + n)}{\Gamma(\lambda + n)} \leqslant A\, n^{-\lambda(1-\cos t)} = A\, e^{-\lambda(1-\cos t)\log n}$$

$$\leqslant A e^{-(1/3)\lambda t^2 \log n}$$

and so

$$\int_0^\delta | \varphi(t)\, K_n(t) | \, dt \leqslant$$

$$\leqslant A \int_0^\delta \frac{| \varphi(t) |}{\sin(t/2)} \,\left| \operatorname{Im} \left\{ \Gamma(\lambda\, e^{it} + n)/\Gamma(\lambda \cos t + n) \right\} \right| e^{-\lambda(1-\cos t)\log n} \, dt$$

$$+ A \int_0^\delta | \varphi(t) | \, e^{-(1/3)\, t^2 \log n} \, dt,$$

the second integral on the right is $O(1/\sqrt{\log n})$; so $o(1)$ as $n \to \infty$.

Finally, from the lemma (cf. n. **2**), we obtain

$$(3.1) \quad \left\{ \begin{aligned} & | S_n^\lambda(x) - f(x) | \leqslant \\ & \leqslant A \int_0^\delta \frac{| \varphi(t) | \; | \sin(\lambda \sin t \log n) |}{\sin(t/2)} \, e^{-\lambda(1-\cos t)\log n} \, dt + o(1) \\ & = O(1) \int_0^\delta \frac{| \varphi(t) |}{t} \, \frac{| \sin(\lambda \sin t \log n) |}{\exp\{\lambda(1-\cos t)\log n\}} \, dt + o(1) \end{aligned} \right. \quad \text{as } n \to \infty.$$

Thus in order to prove the Theorem it remains to show that the integral on the right of (3.1) is $o(1)$ as $n \to \infty$. We set

$$\int_0^\delta \frac{|\varphi(t)|}{t} \frac{|\sin(\lambda \sin t \log n)|}{\exp\{\lambda (1 - \cos t) \log n\}} \, dt =$$

$$= \left[\int_0^{1/\log n} \cdots + \int_{1/\log n}^{1/(\log n)^\alpha} \cdots + \int_{1/(\log n)^\alpha}^{\delta} \cdots \right] = K_1 + K_2 + K_3$$

say, where $0 < \alpha < 1/2$.

From the hypothesis (1.4), we have

$$K_1 = \int_0^{1/\log n} \frac{|\varphi(t)|}{t} \cdot O(\lambda \, t \log n) \, dt = O(\lambda \log n) \int_0^{1/\log n} |\varphi(t)| \, dt =$$

$$= O(\lambda \log n) [o(t/\log(1/t))]_0^{1/\log n} = o(1) , \qquad \text{as } n \to \infty.$$

Next, by the second mean value theorem and the hypothesis (1.4) we have, for $0 < \alpha < \alpha' < 1/2$,

$$K_2 = \int_{1/(\log n)}^{1/(\log n)^\alpha} \frac{|\varphi(t)|}{t} \frac{|\sin(\lambda \sin t \log n)|}{\exp\{\lambda (1 - \cos t) \log n\}} \, dt$$

$$= \frac{1}{\exp\{\lambda \log n \cdot 2 \sin^2(1/(2 \log n))\}} \int_{1/\log n}^{1/(\log n)^{\alpha'}} \frac{|\varphi(t)|}{t} O(1) \, dt = O(1) \int_{1/\log n}^{1/(\log n)^{\alpha'}} \frac{|\varphi(t)|}{t} \, dt$$

$$= O(1) \left[o\left(\frac{1}{\log(1/t)}\right) \right]_{1/\log n}^{1/(\log n)^{\alpha'}} + O(1) \int_{1/\log n}^{1/(\log n)^{\alpha'}} o\left(\frac{1}{t \log(1/t)}\right) dt$$

$$= o\left(\frac{1}{\log \log n}\right) + o\left[\log \log(1/t)\right]_{1/\log n}^{1/(\log n)^{\alpha'}}$$

$$= o(1) + o(\log \alpha') = o(1) , \qquad \text{as } n \to \infty .$$

Lastly, applying the second mean value theorem and by the continuity part of the integral $\int |\varphi(t)|\,dt$, we have, for $1/(\log n)^\alpha < \delta' < \delta$,

$$K_3 = \int_{1/(\log n)^\alpha}^{\delta} \frac{|\varphi(t)|}{t} \frac{|\sin(\lambda \sin t \log n)|}{\exp\{\lambda(1-\cos t)\log n\}}\,dt$$

$$= \frac{(\log n)^\alpha}{\exp\{\lambda \log n \cdot 2\sin^2[1/(2(\log n)^\alpha)]\}} \int_{1/(\log n)^\alpha}^{\delta'} |\varphi(t)|\, O(1)\,dt$$

$$= \frac{(\log n)^\alpha}{\exp(\log n)^{1-2\alpha}} O(1) = o(1) \quad \text{as } n \to \infty, \text{ since } 0 < \alpha < 1/2.$$

This completes the proof of the Theorem.

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