Riv. Mat. Univ. Parma (2) 10 (1969), 71-80

KALYAN DAS GARG (*)

On Heat Transfer in Turbulent Boundary Layer on Flat Plate. (**)

1. - Introduction.

The experiments made by TOWNSEND [1] on the structure of turbulent boundary layer over a smooth plate show that the similarity variable $\xi = y/x$ exists, so that the mean velocity and shearing stress can be expressed as the function of ξ, when x is the distance along the plate, 51 cm. beyond the entrance to the working section of wind tunnel. This similarity property led DAVIES [2] to solve the problem of heat transfer. He solved the problem with the assumptions:

(a) The difference between the temperature of surface of the plate and that of free stream is sufficiently small to prevent the buoyancy effect.

(b) The heat diffusivity is proportional to momentum diffusivity.

His solutions required the knowledge of observed mean velocity and turbulent shearing stress. The same problem was considered by BOURNE and DAVIES [3] with some modifications. In the next paper, BOURNE and DAVIES [4] calculated the rate of heat transfer, when the measured values of mean velocity profile and of eddy viscosity were not available. It is seen that the solution for eddy viscosity ([4], equation (15)) involves complications due to the presence of x.

We now solve the problem of heat transfer by simple technique and our

(*) Indirizzo: p. o. SARMATHURA, Dholpur, Rajasthan, India.
(**) Ricevuto: 12-XII-1968.

assumptions are:

(i) The surface of plate is kept at the temperature T_1 independent of the distance, parallel to the surface, and that of free stream at T_0. Their difference is small enough to prevent the buoyancy effect.

(ii) The formulae for distributions of surface shearing stress, temperature in transition and laminar sub-layers, and for the mean velocity in fully turbulent, transition and laminar sub-layers are available.

(iii) The RENOLDS analogy between heat and momentum transfer is true, and we assume that their dependence on the ξ variable is of same functional form. We take $\varepsilon_H = \varepsilon$, where ε_H is eddy heat diffusivity and ε is eddy momentum diffusivity.

2. - The basic equation of energy.

Adopting the usual notation (x, y) are measured along and normal to the plate with (U, V), the corresponding components of mean velocity in turbulent boundary layer. Denoting the mean temperature by T, and thermal diffusivity by K, when the frictional heat is neglected, the equation of energy for incompressible fluid given by HOWARTH ([5], p. 821) is

$$U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \frac{\partial}{\partial y} \left[K \frac{\partial T}{\partial y} + \varepsilon_H \frac{\partial T}{\partial y} \right]. \tag{1}$$

We see that ε_H is considerable larger than K in the turbulent region, which we can ignore. Now, we can write the basic equation (1) by the assumption (iii) in the form:

$$U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \frac{\partial}{\partial y} \left(\varepsilon \frac{\partial T}{\partial y} \right). \tag{2}$$

But TOWNSEND discovered that his experimental results could be expressed as the function of variable $\xi = y/x$, where x measures distance from a point of 51 cm. beyond the entrance to the working section of wind tunnel. But for our convenience, we take general form $\xi = y/(k\, x^q)$ to obtain the similarity solution, where the parameters k and q are to be determined later on.

3. - Calculation of eddy momentum diffusivity.

Now, the RENOLDS analogy is expressed by the relation

$$\frac{J_y}{\varrho \, \partial U/\partial y} = \frac{q_y}{\varrho \, c_p \, \partial T/\partial y} = \varepsilon , \tag{3}$$

where J_y is shearing stress at normal distance y from the plate, q_y the rate of heat transfer at the same point, ϱ the density of air, and c_p is specific heat at constant pressure.

From the PRANDTL's theory of mixing length, we have

$$\frac{J_y}{\varrho} = l^2 \left(\frac{\partial U}{\partial y}\right)^2 = A^2 \, y^2 \left(\frac{\partial U}{\partial y}\right)^2 . \tag{4}$$

Hence, from (3) and (4),

$$\varepsilon = A^2 \, y^2 \, \frac{\partial U}{\partial u} , \tag{5}$$

and the associate velocity distribution is

$$\frac{U}{U_J} = \frac{1}{A} \log_e \frac{y U_J}{\nu} + B , \tag{6}$$

where U_J is frictional velocity given by the equation

$$\frac{U_J^2}{U_0^2} = 0.03 \left(\frac{\nu}{U_0 \, x}\right)^{1/5} , \tag{7}$$

with the modifications suggested by GOLDSTIEN ([6], p. 362). It is good working formula when RENOLDS number is of order 10^6, where U_0 is free stream velocity and ν is kinematic viscosity.

It is shown by COLES [7] that $A = 0.4$ and $B = 5.10$ for the equation (6) give close approximation in the crucical zone of turbulent boundary layer. Thus, the equation (6) and (7) give close approximation in the region of applicability near the plate surface which is the region of importance for calculating heat transfer from the plate.

It is also seen from ([4], Fig. 1) that COLES's logarithmic formula for velocity distribution given by (6) is in close agreement with the power law formula

$$U = 1.72\ U_0\ \xi^{0.15} \qquad \text{for} \quad \xi \leqslant 0.01$$

between $x = 20$ cm. and 50 cm. when $\xi = y/x^q$, where $q = 0.90$.

Now, the value of ε is obtained by differentiating (6), and putting the value of U_J from (7). It is given by the relation

$$\varepsilon = A\ y\ U_0\ (0.03)^{1/2} \left(\frac{\nu}{U_0\ x}\right)^{1/10}. \tag{8}$$

4. - Similarity solution of heat transfer equation in the fully turbulent flow over flat plate.

Following the REYNOLDS technique of taking long time interval at a point, the equation of continuity for incompressible mean flow is

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{9}$$

and the stream function ψ is taken in the form

$$\psi = x^r f(\xi)\,, \qquad \text{where } \xi = y/(k\ x^q)\,,$$

so that

$$\begin{cases} U = \dfrac{\partial \psi}{\partial y} = k^{-1}\ x^{r-q}\ f'(\xi) \\ V = -\dfrac{\partial \psi}{\partial x} = -r\ x^{r-1} f + q\ x^{r-1}\ \xi\ f'\,, \end{cases} \tag{10}$$

where dashes denotes differentiation with respect to ξ. The eddy momentum diffusivity is given by the relation

$$\varepsilon = A\ k\ x^q\ \xi\ U_0\ (0.03)^{1/2} \left(\frac{\nu}{x\ U_0}\right)^{1/10}. \tag{11}$$

Now the equation (2) becomes

$$(12) \qquad -r\,x^{r-q-1} f\, T' = A\, x^{-q-1/10}\, U_0\, (0.03)^{1/2} \left(\frac{\nu}{U_0}\right)^{1/10} \frac{\partial}{\partial \xi} (\xi\, T') .$$

Condition that (12) may have the solution in ξ only is that $r = 0.90$.

It is seen from the equation (10) that the condition, U may be the function of ξ only is that $r = q = 0.90$.

Now, from the equation (10), we can take

$$(13) \qquad f = (1.72/1.15)\; U_0\, \xi^{1.15} .$$

Since the power law formula gives close approximation, when plotted against the variable $\xi = y/x^{0.90}$, therefore $k = 1$ has been taken in (13).

From the equations (12) and (13), we have

$$-\frac{1.72}{1.15}\; \frac{r\, \xi^{1.15}\, T'}{A\sqrt{0.03}\, (\nu/U_0)^{1/10}} = \frac{\partial}{\partial \xi} (\xi\, T')$$

or

$$-\frac{1.72\, r\, \xi^{0.15}}{1.15\, A\sqrt{0.03}\, (\nu/U_0)^{1/10}} = \frac{\xi\, T'' + T'}{\xi\, T'} ,$$

or

$$(14) \qquad \xi T' = C\, e^{-l_1 \xi^{1.15}}$$

where

$$l_1 = \frac{1.72\, r}{(1.15)^2 A\sqrt{0.03}\, (\nu/U_0)^{1/10}} .$$

Hence

$$(15) \qquad T = C \int_{\xi_*}^{\xi} (1/\xi)\, e^{-l_1 \xi^{1.15}} + D .$$

We have to determine the values of constants C and D in (15) by the boundary conditions:

$$T = T_0 \quad \text{for } \xi \to \infty, \qquad\qquad T = T_* \quad \text{for } \xi = \xi_* ,$$

and $T = T_1$ at the surface of the plate, where the symbol $*$ denotes the value of the variable at the top of the transition layer.

Subject to these boundary conditions, the solution of (15) is

$$(16) \qquad \frac{T - T_*}{T_0 - T_*} = \int_{\xi_*}^{\xi} F(\xi)\, d\xi \Big/ \int_{\xi_*}^{\infty} F(\xi)\, d\xi\,, \qquad \text{where} \quad F(\xi) = (1/\xi)\, e^{-l_1 \xi^{1.15}}.$$

Assuming that the shearing stress and vertical heat flux at a point in the laminar and transition layers are independent of y and are equal to J_0 and q_0 respectively, we have from ([5], p. 823) $q_0 = \varrho\, c_p U_J\, T_r$, where T_r is the friction temperature at the top of transition layer, and this quantity also can be obtained from the equation (16) and it is given by the relation $q_0 = -\,\varrho\, c_p\, \varepsilon \cdot (\partial T/\partial y)$ at $y = y_*$, namely

$$(17) \qquad q_0 = \frac{\varrho\, c_p\, M\, U_0\, (T_* - T_0)}{x^{0.10} \int_{\xi_*}^{\infty} F(\xi)\, d\xi}\,,$$

at the given station from the leading edge, where

$$M = e^{-l_1 \xi_*^{1.15}} A \sqrt{0.03} \left(\frac{\nu}{U_0}\right)^{1/10}.$$

Now,

$$(18) \qquad T_r = \frac{M\, U_0\, (T_* - T_0)}{U_J\, x^{0.10} \int_{\xi_*}^{\infty} F(\xi)\, d\xi}\,.$$

The temperature at a point in the transition layer is given by the form ([5], p. 831)

$$(19) \qquad \frac{T_1 - T}{T_r} = \frac{5}{\lambda} \log_e\!\left(\frac{\lambda\, \sigma\, y_J}{5} + 1 - \lambda\, \sigma\right) + 5\, \sigma\,,$$

where σ is Prandtl number.

It becomes

$$(20) \qquad \frac{T_1 - T_*}{T_r} = \frac{5}{\lambda} \log_e(5\, \lambda\, \sigma + 1) + 5\, \sigma$$

at the point $y = y_*$.

From (18) and (20) we have

$$\frac{T_1 - T_*}{T_0 - T_*} = -\frac{M\,U_0}{U_J\,x^{0.10}}\;\frac{(5/\lambda)\log_e(5\lambda\sigma + 1) + 5\sigma}{\int_{\xi_*}^{\infty} F(\xi)\,d\xi}. \tag{21}$$

Now,

$$\theta(\xi) = \frac{T - T_1}{T_0 - T_1} = \left(\frac{T - T_*}{T_0 - T_*} - \frac{T_1 - T_*}{T_0 - T_*}\right) \Big/ \left(1 - \frac{T_1 - T_*}{T_0 - T_*}\right) = \frac{\Omega_1}{\Omega_2}, \tag{22}$$

with

$$\Omega_1 = \int_{\xi_*}^{\xi} F(\xi)\,d\xi + \frac{MU_0\,\{(5/\lambda)\;\log_e(5\lambda\sigma + 1) + 5\sigma\}}{U_J\,x^{0.10}\int_{\xi_*}^{\infty} F(\xi)\,d\xi},$$

$$\Omega_2 = 1 + \frac{MU_0\,\{(5/\lambda)\;\log_e(5\lambda\sigma + 1) + 5\sigma\}}{U_J\,x^{0.10}\int_{\xi_*}^{\infty} F(\xi)\,d\xi}.$$

At $x = 50$, $\lambda = 1$, $\sigma = 0.72$, $U_0 = 35$ m./sec., the θ-distribution is given by the equation

$$\theta(\xi) = 1 + 0.1142\,E(-l_1\,\xi^{1.15}), \tag{23}$$

where

$$E(-l_1\,\xi^{1.15}) = -\int_{l_1\xi^{1.15}}^{\infty} (e^{-t}/t)\,dt, \qquad l_1 = 46.64.$$

Table

y (cm.)	0.2	0.4	0.6	0.8	1.0	1.2
θ	0.78975	0.86794	0.90839	0.93601	0.95015	0.96373

(Temperature distribution at $x = 50$ cm. from the leading edge of the plate, when $U_0 = 35$ m./sec.).

An expression for the mean rate of transfer of heat in fully turbulent region per unit widt of the plane section perpendicular to the plate may be obtained by the expression

$$\varrho\, c_p \int_{y_*}^{\infty} U\cdot(T-T_0)\,\mathrm{d}y \qquad \text{at the given station } x=x_1\,.$$

From the equations (10) and (23), it can be written in the form

$$(24)\qquad -1.72\times 0.1142\,\varrho\, c_p\, U_0\, x_1^q\,(T_1-T_0)\int_{\xi_*}^{\infty} E(-l_1\,\xi^{1\cdot 15})\,\xi^{0\cdot 15}\,\mathrm{d}\xi\,.$$

The similarity solution of equation (2) in the transition and laminar layer does not exist in the terms of variable ξ. We adopt an approximate method.

5. - An approximate method of heat transfer problem in the laminar and transition layers of the flow over the smooth flat plate.

The mean velocity distributions very near the flat plate ([5], p. 824) are of the forms

$$(25)\qquad \frac{U}{U_J}=\frac{U_J\,y}{\nu}=y_J \qquad (y_J\leqslant 5)$$

in the laminar sub-layer and

$$(26)\qquad \frac{U}{U_J}=-3.05+5\,\log_e y_J \qquad (5\leqslant y_J\leqslant 30)$$

in transition layer.

An expression for the flux of heat through the region from $y=0$ to $y=y_*$ across unit width of the plane section normal to the plate at a distance $x=50$ cm. is obtained from empirical velocity and temperature forms ([5], p. 831). We find that the contribution to the flux in the transition zone, from $y=5$ to $y=30$ is given at $x=50$ cm. by the expression:

$$(27)\qquad \varrho\, c_p \int_{5\nu/U_J}^{30\nu/U_J} U\cdot(T-T_0)\,\mathrm{d}y=\varrho\, c_p\,(T_1-T_0)\,U_J\int_{5\nu/U_J}^{30\nu/U_J}[-3.05+5\log_e y_J]\,.$$

$$\cdot\left[1-\frac{MU_0\{(5/\lambda)\log_e(\frac{1}{5}\,\lambda\,\sigma\,y_J+1-\lambda\,\sigma)+5\sigma\}}{H}\right]\,dy\,,$$

with

$$H=U_J\,x^{0\cdot10}\int\limits_{\xi_*}^{\infty}F(\xi)\,d\xi+MU_0\{(5/\lambda)\log_e(5\lambda\,\sigma+1)+5\sigma\}.$$

The contribution to the flux by laminar sub-layer is given by the expression

$$(T_1-T_0)\,\varrho\,c_p\int\limits_0^{5\nu/U_J}y_J\left[1-\frac{MU_0\,\sigma\,y_J}{H}\right]\,dy\,. \tag{28}$$

6. - The comparision of theoretical and experimental evaluations.

TOWNSEND shows that his results over the plane wall are function of y/x, x being measured from a point 51 cm. beyond the entrance to the wall. We assume that the effect of presence of completely laminar layer, near the leading edge of the plate may be allowed by similarity variable $\xi=y/x^{0.90}$, where x measures distance from the leading edge of the plate. We have taken the example associated with the highest free stream velocity (35 m./sec.), used by ÉLIÁS [8], so that the complicated effect due to the presence of completely laminar region near the leading edge is as small as possible, We find that y/x^q, where $q=0.90$ could be used to provide reasonably satisfactory similarity variable.

Taking appropriate values of ϱ, c_p, ν, ξ_* and $\sigma=0.72$, $\lambda=1$, we compute θ-distribution and the rate of heat transfer for the length $x=50$ cm. from the leading edge. When we plot θ-distribution against the variable y, it is seen that there is no significant difference between theoretical and measured results.

We also find that the net rate of heat transfer Q is obtained by adding expressions (24), (27) and (28). We obtain

$$\frac{Q}{T_1-T_0}=0.55010+0.02407+0.00189$$

or $Q/(T_1-T_0)=0.57606$ Watts per degree centigrade against 0.56 Watts per degree centigrade measured by ÉLIÁS.

A c k n o w l e d g e m e n t . I express my gratitude to Dr. J. P. AGARWAL (Head of the Department of Mathematics, Agra College, Agra) for his guidance.

References.

[1] A. A. Townsend, *The structure of the turbulent boundary layer*, Proc. Cambridge Philos. Soc. **47** (1951), 375-391.

[2] D. R. Davies, *Heat transfer from a flat plate through a turbulent boundary*, Quart. J. Mech. Appl. Math. **8** (1955), 326-337.

[3] D. R. Davies and D. E. Bourne, *On the calculation of heat and mass transfer in laminar and turbulent boundary layers* (II): *The turbulent case*, Quart. J. Mech. Appl. Math. **9** (1956), 468-488.

[4] D. E. Bourne and D. R. Davies, *On the calculation of eddy viscosity and heat transfer in a turbulent boundary layer on a flat surface*, Quart. J. Mech. Appl. Math. **11** (1958), 223-234.

[5] L. Howarth (Editor), *Modern Developments in Fluid Dynamics. High Speed Flow.* Clarendon Press, Oxford 1953.

[6] S. Goldstien (Editor), *Modern Developments in Fluid Dynamics*, Clarendon Press, Oxford 1938.

[7] D. Coles, *The law of the wake in the turbulent boundary layer*, J. Fluid. Mech. **1** (1956), 191-226.

[8] F. Éliás, *Die Wärmeübertragung einer geheizten Platte an strömende Luft* (II): *Vergleich der Versuchsergebnisse mit der Theorie*, Z. angew. Math. **10** (1930), 1-14.

Abstract.

In this paper we have considered in forced convection the problem of heat transfer from the surface of heated flat plate to air stream by the use of similarity property, suggested by Townsend [1]. *It is found that if logarithmic formula for velocity distribution is represented by power low formula in the terms of similarity variable ξ only, the problem of heat transfer can be solved by following the Renolds analogy between heat and momentum transfer* [(5], p. 819).

We take the value of eddy momentum diffusivity from the Prandtl's theory of mixing length. Assuming the validity of empirical formulae for distributions of surface shearing stress, temperature in transition and laminar sub-layers and for the mean velocity in the fully turbulent, transition and laminar sub-layers, the problem is completely solved. When, we compare the theoretical results with experimental results obtained by Éliás [8], *it is seen that they are in close agreement.*

* * *