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On Commuting Functions and Fixed Points. (**)

ELDON DYER in 1954, ALLEN SHIELDS in 1955 and LESTER DUBINS in 1956 independently conjectured that if f and g are continuous functions which map a closed interval of the real line into itself and which commute, then they have a common fixed point. This has been disproved by BOYCE [1] and HUNEKE [3] independently.

ALLEN SHIELDS [7] proved that if f and g map the unit disc $|z| \leqslant 1$ in the complex plane into itself in a continuous manner, if they are analytic in the open disc and if they commute $[fg(z) = gf(z)$ for all $z]$, then they have a common fixed point $[f(z_0) = z_0 = g(z_0)]$. More generally, any commuting family of such functions has a common fixed point.

We prove the following theorem for linear functions which commute:

Theorem 1. *If* $f(z) = az + b$, $a \neq 1$, *then* $g(z) = cz + d$ *commutes with* f *if and only if they have a common fixed point.*

In order to prove the Theorem we need the following results:

Lemma. *Let* f *and* g *be linear functions in the complex plane. Then* f *and* g *commute if and only if* $fg(0) = gf(0)$.

Proof. Let $f(z) = az + b$ and $g(z) = cz + d$. $fg = gf$ if and only if $f(g(z)) = g(f(z))$. But

$$f(g(z)) = f(cz + d) = a\,(cz + d) + b = acz + ad + b\,,$$

and

$$g(f(z)) = g(az + b) = c\,(az + b) + d = acz + bc + d\,.$$

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Therefore, $fg = gf$ if and only if $ad + b = bc + d$. Now,

$$f(g(0)) = f(d) = ad + b\,, \qquad g(f(0)) = g(b) = bc + d\,.$$

Hence $fg = gf$ if and only if

$$f(g(0)) = g(f(0))\,.$$

Corollary. *If $f(z) = az + b$ and $g(z) = cz + d$, then $fg = gf$ if and only if $f(d) = g(b)$.*

Definition. A point z_0 which is invariant under a transformation is called « a fixed point of the transformation ».

The linear functions of the form $f(z) = az + b$, $a \neq 1$, have unique fixed points. The linear function $f(z) = z$ fixes all points. These are the only linear functions with fixed points. The only linear functions which have no fixed points are those of the form $f(z) = z + b$, $b \neq 0$.

The identity function $f(z) = z$ commutes with every function.

Proof of Theorem 1. Let f and g commute and z_0 be the unique fixed point of f. Then we prove that z_0 is a fixed point of g. $fg = gf$ implies that $f(g(z_0)) = g(g(z_0))$. Now $f(g(z_0)) = g(z_0)$, since $f(z_0) = z_0$.

This gives $g(z_0)$ a fixed point of f. But the unique fixed point of f is z_0. Therefore, $g(z_0) = z_0$.

Thus f and g have a common fixed point.

Conversely, let f and g have a common fixed point. We want to prove that f and g commute.

The unique fixed point of f is $b/(1 - a)$. Therefore $z_0 = b/(1 - a)$. Since $z_0 = g(z_0) = cz_0 + d$, we have $d = (1 - c)\, z_0 = (1 - c)\, b/(1 - a)$. This gives $ad + b = cb + d$. But $ad + b = f(d)$ and $cb + d = g(b)$. So that $f(d) = g(b)$. Hence by Corollary, we get $fg = gf$.

Thus the Theorem.

A similar result in real number system has been given by SEGUIN [6].

KAKUTANI [4] and MARKOV [5] have shown that if a commutative family of continuous linear transformations of a linear topological space into itself leaves some non empty compact convex subset invariant, then the family has a common fixed point. DE MARR [2] proved that if B is a BANACH space and is a non empty compact convex subset of B and if T is a commutative family of contraction mapping of into itself, then the family T has a common fixed point.

Here we prove the following theorem. The proof is very simple but the result seems to be new and interesting.

Theorem 2. *Let E be a complete metric space. If f and g are two contraction mappings of E into itself and if they commute, then they have a common fixed point.*

In the proof of this theorem we will make use of BANACH contraction principle, which is stated in the following form:

« If T is a contraction mapping of a complete metric space E into itself, then T has a unique fixed point. »

Proof. The functions f and g are commuting contraction mappings of E into itself, i. e.

$$fg(x) = gf(x) \qquad \text{for all } x \text{ in } E.$$

By the contraction mapping principle, we know that f and g have unique fixed points. Let x_0 be the unique fixed point of f. Then

$$fg(x_0) = gf(x_0) = g(x_0) .$$

Thus $g(x_0)$ is a fixed point of f. But f has a unique fixed point. Therefore $g(x_0) = x_0$.

Hence the proof.

Remarks. (1) If f is a contraction and g is a mapping commuting with f, then they have a common fixed point.

(2) The theorem will remain true for a family of commutative contraction mappings.

References.

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