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Some Further Extensions of Banach's Contraction Principle. (**)

Introduction.

The most elementary and by far the most fruitful method for proving theorems on the existence and uniqueness of solutions is the principle formulated by BANACH ([1]) and first applied to the proof of an existence theorem by CACCIOPPOLI ([2]). For this purpose, extensions of the theorem are of continuing interest. In the present paper few extensions of « BANACH's Contraction Principle » have been discussed in detail. A theorem related to the converse of BANACH's contraction principle is also added.

1.1. – Let X be any set. A function d from $X \times X$ into non negative real numbers is called a pseudometric on X if it satisfies the following requirements:

(1) $d(x, y) \geqslant 0$ for all x, y;

(2) if $x = y$, then $d(x, y) = 0$;

(3) $d(x, y) = d(y, x)$ for all x, y;

(4) $d(x, y) \leqslant d(x, z) + d(z, y)$ for every triple of points.

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([1]) S. BANACH, *Sur les operations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133-181.

([2]) R. CACCIOPPOLI, *Un teorema generale sull'esistenza di elementi in una trasformazione funzionale*, Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. **11** (1930), 794-799.

The function d is called a metric on X if the condition (2) is replaced by the following one:

$$(2^*) \qquad d(x, y) = 0 \qquad \text{if } x = y.$$

1.2. – A metric space X is said to be ε-chainable or well-linked if for every pair (a, b) of points of X and for every $\varepsilon > 0$ there exists a finite sequence a_1, a_2, ..., a_n of points, of X, with $a = a_1$, ..., and $a_n = b$, such that $d(a_i, a_{i+1}) < \varepsilon$ for every $i < n$. In other words, a and b can be joined by a chain of steps at most equal to ε.

Remark. Every connected metric space is well-linked but the converse is not true. For example the set Q of rationals is well-linked but not connected. However, the converse holds if X is compact.

2.1. – A mapping T of a metric space X into itself is said to satisfy a LIPSCHITZ condition with LIPSCHITZ constant λ if

$$d\big(T(p), T(q)\big) \leqslant \lambda\, d(p, q) \qquad (p, q \in X).$$

If this condition is satisfied with a LIPSCHITZ constant λ such that $0 \leqslant \lambda < 1$, then T is called a contraction mapping.

2.2. – A mapping T of X into itself is said to be locally contractive if for every $x \in X$ there exist ε and λ ($\varepsilon > 0$, $0 < \lambda < 1$) which may depend on x such that

$$p, q \in S(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\} \implies$$

$$\implies d(T(p), T(q)) < \lambda\, d(p, q) \qquad (p, q \in X;\ p \neq q).$$

2.3. – A mapping T of X into itself is said to be (ε, λ)-uniformly locally contractive if it is locally contractive and ε and λ do not depend on x.

2.4. – A mapping T: $X \to X$ of a metric space X into itself is said to be nonexpansive (ε-nonexpansive) if the condition

$$d\big(T(p), T(q)\big) \leqslant d(p, q),$$

holds for all $p, q \in X$ [for all p, q with $d(p, q) < \varepsilon$]. If we have strict inequality sign for all $p, q \in X$, $p \neq q$ (for all $p, q \in X$ such that $0 < d(p, q) < \varepsilon$), then T is said to be contractive (or ε-contractive).

Remark. The assumption $d(T(p), T(q)) < d(p, q)$ is not sufficient for the existence of a fixed point. For example if T is a mapping of R_+ into itself defined by

$$T(x) = \sqrt{x^2 + 1}\,,$$

then T does not have a fixed point.

3.1. – Theorem. *Let T be ε-contractive mapping of an ε-chainable metric space X into itself, i. e.*

$$0 < d(x, y) < \varepsilon \qquad \Longrightarrow \qquad d(T(x), T(y)) < d(x, y)\,, \qquad \forall\, x, y \in X\,,$$

$x \neq y$, *satisfying*

$$(\exists\, x \in X): \quad \{T^n(x)\} \supset \{T^{n_i}(x)\} \qquad \textit{with} \ \lim_{i \to \infty} T^{n_i}(x) \in X\,.$$

Then T has a unique fixed point.

Proof. Since (X, d) is ε-chainable we define, for every x, y in X,

$$d_\varepsilon(x, y) = \inf \sum_{i=1}^{n} d(x_{i-1}, x_i)\,,$$

where the infimum is taken over all ε-chains $x_0, x_1, x_2, \ldots, x_n$ joining $x = x_0$ and $y = y_n$. Thus d is a distance function on X satisfying:

$$(1) \qquad d(x, y) \leqslant d_\varepsilon(x, y)\,,$$

$$(2) \qquad d(x, y) = d_\varepsilon(x, y) \qquad \text{for} \quad d(x, y) < \varepsilon.$$

From (2) it follows that a sequence $\{x_n\}$ is a Cauchy sequence with respect to d_ε if it is a Cauchy sequence with respect to d and is convergent with respect to d_ε if and only if it converges with respect to d. Therefore, since (X, d) is complete (X, d_ε) is also a complete metric space. Moreover, T is ε-contractive mapping with respect to d_ε. Given $x, y \in X$ $(x \neq y)$ and any ε-chain $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = x$ and $x_n = y$, we have

$$d(x_i, x_{i-1}) < \varepsilon \qquad (i = 1, 2, \ldots, n).$$

So that

$$d(Tx_{i-1}, Tx_i) < d(x_{i-1}, x_i) < \varepsilon \qquad (i = 1, 2, \ldots, n).$$

Hence $Tx_0, Tx_1, \ldots, Tx_n$ is an ε-chain joining Tx and Ty and

$$d_\varepsilon\big(T(x), T(y)\big) \leqslant \sum_{i=1}^{n} d\big(T(x_{i-1}), T(x_i)\big) < \sum_{i=1}^{n} d(x_{i-1}, x_i),$$

$x_0, x_1, x_2, \ldots, x_n$ being an arbitrary ε-chain, we have

$$d_\varepsilon\big(T(x), T(y)\big) < d_\varepsilon(x, y).$$

Thus T is contractive with respect to d_ε. We have already proved that d_ε is a complete metric space. Hence, by theorem 1 of [8] has a unique fixed point.

3.2. - Theorem. *Let T be an (ε, λ)-uniformly locally contractive mapping of a complete ε-chainable pseudometric space X into itself. Then T has a unique fixed point.*

Proof. Let a be an arbitrary point of X. Consider the ε-chain:

$$a = a_0, a_1, a_2, \ldots, a_k = T(a).$$

By the triangle inequality

$$d\big(a, T(a)\big) \leqslant \sum_{i=1}^{k} d(a_{i-1}, a_i) < k\varepsilon. \tag{1}$$

For a pair of consecutive points of the ε-chain the condition $p, q \in S(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\} \implies d\big(T(p), T(q)\big) < \lambda\, d(p, q)$, $p \neq q$, is satisfied.

Let a be an arbitrary point in X. Now set $a_1 = Ta$, $a_2 = T^2 a$ and in general let $a_n = Ta_{n-1} = T^n a$. We shall show that the sequence $\{a_n\}$ is CAUCHY. In fact, we have

$$\begin{aligned} d(a_n, a_m) &= d\big(T^n(a), T^m(a)\big) < \lambda^n d(a, a_{n-m}) \\ &< \lambda^n\{d(a, a_1) + d(a_1, a_2) + \ldots + d(a_{m-n-1}, a_{m-n}\} \\ &< \lambda^n d(a, a_1)\{1 + \lambda + \lambda^2 + \ldots + \lambda^{m-n-1}\} \\ &< \frac{\lambda^n}{1-\lambda}\, d(a, a_1) \\ &< \frac{\lambda^n}{1-\lambda}\, d\big(a, T(a)\big) \\ &< \frac{\lambda^n}{1-\lambda}\, k_\varepsilon, \qquad \text{from (1).} \end{aligned}$$

Since $\lambda < 1$ this quantity is arbitrarily small for sufficiently large n. Thus $\{a_n\}$ is a CAUCHY sequence. Since X is complete $\lim_{n\to\infty} a_n$ exists. We set $a_0 = \lim_{n\to\infty} a_n$, then by virtue of continuity of T, $T(a_0) = T \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = a_0$. Thus T has a fixed point a_0, i. e. $T(a_0) = a_0$.

In order to complete the proof we have to show that $a_0 = \lim_{n\to\infty} a_n$ is a unique fixed point satisfying $T(a_0) = a_0$.

Let a_0 and b_0 be two different fixed points, i. e. $T(a_0) = a_0$ and $T(b_0) = b_0$, $a_0 \neq b_0$. Then $a_0 \neq b_0$ implies $d(a_0, b_0) > 0$.

Now

$$d\big(T(a_0), T(b_0)\big) = \lim d\big(T(a_n), T(b_n)\big) = d\big(T(a_n), T(b_n)\big) \qquad \text{when } n \to \infty,$$

hence

$$d\big(T(a_0), T(b_0)\big) \leqslant \frac{\lambda^n}{1-\lambda}\, d\big(a_0, T(b_0)\big). \tag{2}$$

Let a be an arbitrary point of X. Consider the chain $a_0 = a_1, a_2, \ldots, a_{k+1} = b = T(b_0)$. Then by the triangle inequality

$$d\big(a_0, T(b_0)\big) \leqslant \sum_{i=1}^{k} d(a_i, a_{i+1}) < k_\varepsilon. \tag{3}$$

Therefore, by (3) equation, (2) becomes

$$d\big(T(a_0), T(b_0)\big) < \frac{\lambda^n}{1-\lambda} k_\varepsilon \to 0 \qquad \text{as } n \to \infty.$$

Thus $d(a_0, b_0) = 0$, which is impossible unless $a_0 = b_0$. Therefore a_0 is a unique fixed point for T.

4.1. – A subset M of a metric space (X, d) is a straight set, provided, for every three points x, y and z in M at least one of the following holds:

$$d(x, y) + d(y, z) = d(x, z), \tag{1}$$

$$d(x, z) + d(z, y) = d(x, y), \tag{2}$$

$$d(y, x) + d(x, z) = d(y, z). \tag{3}$$

A straight set is of Type I, provided, it has only four members, i.e. $M = \{x, y, z, t\}$, and $d(x, y) = d(z, t) = m$, $d(x, t) = d(y, z) = n$, and $d(x, z) = d(y, t) = m + n$, where m and n are positive real numbers. A straight set is of Type II if and only if is not of Type I.

4.2. – Theorem. *Let T be (ε, λ)-uniformly locally contractive mapping of a straight set of Type* II. *Then T is globally contractive with the same λ.*

Proof. A straight set of Type II together with a, b also contains a metric segment whose extremities are a and b that is a subset isometric with a subset of length $d(a, b)$ in E^k. Also the straight set of Type II are metrically equivalent to subset of the real line.

Using these facts, we have, if $p, q \in M$, then there are points $x_0, x_1, x_2, \ldots, x_n$ with $a = x_0$ and $b = x_n$ such that

$$d(a, b) < \sum_{i=1}^{n} d(x_{i-1}, x_i) \qquad \text{and} \qquad d(x_{i-1}, x_i) < \varepsilon_0 .$$

Therefore,

$$d(T(a), T(b)) \leqslant \sum_{i=1}^{n} d(T(x_{i-1}), T(x_i))$$

$$< \lambda \sum_{i=1}^{n} d(x_{i-1}, x_i) = \lambda\, d(a, b) .$$

Hence the theorem.

The above theorem is a generalization of a theorem by M. Edelstein [7] in which he has taken X as a convex and complete metric space. Thereby we have the following

Corollary. *If X is a convex complete metric space and T is an (ε, λ)-uniformly locally contractive mapping of X into itself, then T has a unique fixed point.*

Proof. By proposition 1 [7] T becomes a globally contractive mapping of X into itself. Therefore, by Banach's contraction principle T has a unique fixed point.

5.1. – Denote by F the family of all functions $\lambda(x, y)$ satisfying the following conditions:

(1) $\lambda(x, y) = \lambda(d(x, y))$, i.e. λ is dependent only on the distance x and y.

(2) $0 \leqslant \lambda(d) < 1$ for all $d > 0$.

(3) $\lambda(d)$ is monotonically decreasing function of d.

Let (X, d) be a complete metric space and T a contractive mapping of X into itself satisfying

$$d(T(x), T(y)) \leqslant \lambda(d(x, y))\, d(x, y),$$

where $x, y \in X$ and $\lambda(x, y) \in F$. Then it follows by a theorem by RAKOTCH [15] that the iterated images $T^n(x)$ of X shrinks to the point ξ of X. This can be written in the form

$$\bigcap_{n=1}^{\infty} T^n(X) = \{\xi\}.$$

Since this formula does not involve a metric and has a topological character, it is natural to ask the following question:

Let X be a compact metrizable topological space and T a continuous mapping of X into itself which has the property that $\cap\, T^n(X) = \{\xi\}$. Is it possible to find a metric $d^*(x, y)$ generating the given topology of X such that the mapping T is contractive with respect to d^*?

The question was answered in the affirmative in [16].

Let P be a subspace of metrizable topological space X. Let R denote the set of all contractive mappings of P into itself satisfying $d(Tx, Ty) \leqslant \lambda(x, y) \cdot d(x, y)$, where $x, y \in P$ and $\lambda(x, y) \in F$, and S denote the set of all continuous mappings of P into itself such that $\cap\, T^n(P) = \{\xi\}$ a singleton.

5.2. – Theorem. *A metrizable space M is compact if $R = S$ for every nonempty closed subset P of M.*

Proof. We will prove the theorem by contradiction. Let us assume that M is not pre-compact, and d^* is the metric on M generating the given topology on M. Then there exists an ε and an infinite sequence $\{x_n\}$ such that $d^*(x_m, x_n) > \varepsilon$ for $m \neq n$. Therefore the nonempty set $P = \{x_n\}$ is closed. Now we define T as follows:

$$T(x_0) = x_0 \qquad \text{and} \qquad T(x_n) = x_{n+1}.$$

This mapping for every positive integer n belongs to R, and is contractive on P, by hypothesis. Assuming d as the metric on P associated with T we will prove that $\{x_n\}$ is a CAUCHY sequence with respect to d. Therefore, $\{x_n\}$ will also be a CAUCHY sequence with respect to d^*.

We will first prove that $\{x_n\}$ is bounded. Assume $T(x_0) \neq x_0$ and let

$$x_{n+1} = T(x_n) \qquad (n = 0, 1, 2, \ldots). \tag{1}$$

Since T is a contractive mapping, the sequence $d(x_n, x_{n+1})$ is by (1) non-increasing. And also by the assumption $T(x_0) \neq x_0$, it follows that

$$d(x_n, x_{n+1}) < d(x_0, x_1) \qquad (n = 1, 2, 3, \ldots). \tag{2}$$

Again by the triangle inequality

$$d(x_0, x_n) \leqslant d(x_0, x_1) + d(x_1, x_{n+1}) + d(x_n, x_{n+1}). \tag{3}$$

And by the help of (2) we can write (3) as

$$d(x_0, x_n) < d(x_0, x_1) + d(x_0, x_1) + d(x_1, x_{n+1}),$$

or

$$d(x_0, x_n) < 2\, d(x_0, x_1) + d(x_1, x_{n+1}),$$

or

$$d(x_0, x_n) - d(x_1, x_{n+1}) < 2\, d(x_0, x_1). \tag{4}$$

Now by the definition of T

$$d(x_1, x_{n+1}) = d\big(T(x_0), T(x_n)\big) \leqslant \lambda\big(d(x_0, x_n)\big)\, d(x_0, x_n). \tag{5}$$

Hence, by (4) and (5),

$$d(x_0, x_n) - \lambda\big(d(x_0, x_n)\big)\, d(x_0, x_n) < 2\, d(x_0, x_1),$$

or

$$\big\{1 - \lambda\big(d(x_0, x_n)\big)\big\}\, d(x_0, x_n) < 2\, d(x_0, x_1),$$

or

$$d(x_0, x_n) < \frac{2\, d(x_0, x_1)}{1 - \lambda\big(d(x_0, x_n)\big)}.$$

Now if $d(x_0, x_n) \geqslant d_0$ for a given $d_0 > 0$, then by the monotonicity of $\lambda(d)$ it follows that $\lambda(d(x_0, x_n)) \leqslant \lambda(d_0)$ and therefore

$$d(x_0, x_n) < \frac{2\, d(x_0, x_1)}{1 - \lambda(d(x_0, x_n))} \leqslant \frac{2\, d(x_0, x_1)}{1 - \lambda(d_0)} = C .$$

Hence

$$d(x_0, x_n) \leqslant R \qquad (n = 1, 2, \ldots), \tag{6}$$

where $R = \max(d_0, C)$. Thus the sequence is bounded.

Now for $j > 0$, where j is any positive integer, we have, by definition of T,

$$d(x_{l+1}, x_{l+j+1}) \leqslant \lambda(x_l, x_{l+j})\, d(x_l, x_{l+j}) .$$

Taking the product from $l = 0$ to $l = n-1$ and by dividing both sides by the same terms, we have

$$d(x, x_{n+j}) \leqslant d(x_0, x_j) \prod_{l=1}^{n-1} \lambda(x_l, x_{l+j}) ,$$

which by (6) reduces to

$$d(x_n, x_{n+j}) \leqslant R \prod_{l=1}^{n-1} \lambda(x_l, x_{l+j}) . \tag{7}$$

Now we will show that given $\varepsilon > 0$, there exists a number N depending on ε only not on j, such that for every $j > 0$ there is $d(x_N, x_{N+j}) < \varepsilon$.

If $d(x_l, x_{l+j}) \geqslant \varepsilon$ for $l = 0, 1, 2, \ldots, n-1$, then by definition of T and monocity of $\lambda(d)$ we have $\lambda(x_l, x_{l+j}) = \lambda(d(x_l, x_{l+j})) \leqslant \lambda(\varepsilon)$ and by (7) we have

$$d(x_n, x_{n+j}) \leqslant R\, [\lambda(\varepsilon)]^n .$$

But $\lambda(\varepsilon) < 1$, and $[\lambda(\varepsilon)]^n \to 0$ as $n \to \infty$. Therefore, there exists a positive integer N, not depending on j, such that $d(x_N, x_{N+j}) < \varepsilon$ for every $j > 0$. This proves that $\{x_n\}$ is CAUCHY sequence. Hence the theorem.

As an application of the foregoing material we prove the generalization of the following theorem given by DEMARR [6].

Theorem. If f and g are two mappings of closed unit interval into itself, such that $fg(x) = gf(x)$ for all $x \in I$, $|f(x) - f(y)| \leqslant \alpha |x - y|$ and $|g(x) - g(y)| \leqslant \beta |x - y|$, where β is any positive real number and $0 \leqslant \alpha < 1$, then f and g have a common fixed point.

6.1. – Theorem. *If f and g are two mappings of a closed unit interval into itself, such that $fg(x) = gf(x)$ for all $x \in I$, $|f(x) - f(y)| \leqslant \alpha |x - y|$ and g is any continuous function, then f and g have a common fixed point, provided* $0 < \alpha < 1$.

Proof. Since a closed subset of a complete metric space is complete Therefore, a closed unit interval being a subset of R is also a complete metric space. The condition $|f(x) - f(y)| \leqslant \alpha |x - y|$ for x, y in the closed unit interval where $0 \leqslant \alpha < 1$ implies that f is a contraction mapping. Thus f is a contraction mapping of a complete metric space into itself. Therefore by BANACH's contraction principle f has a unique fixed point in the closed unit interval: i.e., there exists a unique point x_0 in I such that $f(x_0) = x_0$.

Now it is given that

$$fg(x) = gf(x) \qquad \text{for all } x \text{ in } I.$$

Therefore, $fg(x_0) = gf(x_0) = g[f(x_0)] = g(x_0)$. Thus $g(x_0)$ is a fixed point for f. But f has a unique fixed point, say x_0. Therefore $g(x_0) = x_0$ and thus x_0 is a fixed point for g. Hence the theorem.

7.1. – A continuous mapping T of a metric space X into itself is eventually contractive if for every distinct pair x, $y \in X$, there exists $n(x, y)$ a member of I^+ (the positive integers), such that

$$d(T^n(x), T^n(y)) < d(x, y). \tag{1}$$

7.2. – x is proximal to y under T if for each $\alpha > 0$ there exists n, a member of I^+, such that

$$d\big(T^n(x), T^n(y)\big) < \alpha.$$

If x and y are not proximal under T they are said to be distal under T.

7.3. – Theorem. *If T satisfies* (1), *and K is a homeomorphism of X onto X, a compact metric space X into itself, then $K\,T\,K^{-1}$ satisfies* (1). *In addition T has a unique fixed point.*

Proof. By theorem 1.3 [**1**], $K^{-1}x$ and $K^{-1}y$ are proximal. Also since K is a homeomorphism, therefore K and K^{-1} both are continuous. Since K^{-1} is continuous and X is compact, there exists $\delta > 0$ such that $d(w, z) < \delta$ implies $d(K^{-1}(w), K^{-1}(z)) < d(x, y)$. Now $K^{-1}(x)$ and $K^{-1}(y)$ are proximal under T, therefore for each $\delta > 0$ there exists n, a member of I^+, such that $d(T^n(K^{-1}(x)), T^n(K^{-1}(y))) < \delta$. Hence $d((K\,T\,K^{-1})^n(x), (K\,T\,K^{-1})^n(y)) = d(K\,T^n\,K^{-1}(x), K\,T^n\,K^{-1}(y)) < d(x, y)$.

Again by theorem 1.3 [**1**], x and $K\,T\,K^{-1}(x)$ are proximal under T. Now choose $\{n_i\} \in I^+$ such that $n_i < n_{i+1}$ and $d((K\,T\,K^{-1})^{n_i}(x), (K\,T\,K^{-1})^{n_i}(y)) < 1/i$.

By compactness of X we may assume that $\{(K\,T\,K^{-1})^{n_i}(x)\} \to \xi$, and $\{(K\,T\,K^{-1})^{n_i}(y)\} \to \eta$, for some ξ and η in X. Clearly $\xi = \eta$. Also the continuity of $K\,T\,K^{-1}$ implies $K\,T\,K^{-1}\,\eta = \eta$, so that η is a fixed point of $K\,T\,K^{-1}$. That this point is unique is immediate.

Since $K\,T\,K^{-1}$ has a unique fixed point η, $K\,T\,K^{-1}\,\eta = \eta$, or $K^{-1}\,K\,T\,K^{-1}(\eta) = K^{-1}\,\eta$ or $T(K^{-1}\,\eta) = K^{-1}\,\eta$. Thus $K^{-1}\,\eta$ is a unique fixed point of T.

8.1. – Given a vector space E, a norm on E is a map $x \to \|x\|$ from E into the set R_+ of positive real numbers which satisfies the following axioms.

(1) $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|\,\|x\|$ for all $\lambda \in K$ and $x \in E$, where K is the field of real or complex numbers.

(3) $\|x + y\| \leqslant \|x\| + \|y\|$ (the triangle inequality).

A vector space on which a norm is defined is called a normed vectorspace or simply a normed space.

A normed vectorspace E is called a BANACH space if it is complete as a metric space

8.2. – Let E be a normed linear space. A mapping T of E into itself is called a contracting mapping if, for all $x, y \in E$,

$$\|T(x) - T(y)\| \leqslant \alpha \|x - y\|, \qquad \text{where } 0 < \alpha < 1.$$

8.3. – A mapping T of a normed linear space E into itself is called a non-expanding mapping if, for all $x, y \in E$,

$$\|T(x) - T(y)\| \leqslant \|x - y\| .$$

BERS ([2], page 81) has given the following

Theorem. Let $K = \{x \mid \|x\| \leqslant 1\}$ be a subset of a BANACH space and let T be a continuous contracting mapping of K into itself. Then T has one and only one fixed point.

Here we would like to remark that the above theorem may be put in even general set as follows. At the same time the condition of continuity is superfluous.

8.4. – Theorem. *Let K be a closed and convex subset of a finite dimensional Banach space and let T be a contracting mapping of K into itself. Then T has one and only one fixed point.*

Proof. Let x_0 be an arbitrary point in K and consider the sequence $\{x_n\}$. Let $x_1 = T(x_0)$, $x_2 = T(x_1)$, $\ldots$. Now

$$\|x_{n+1} - x_n\| = \|T(x_n) - T(x_{n-1})\| \leqslant \alpha \|x_n - x_{n-1}\| ,$$

and

$$\|x_n - x_{n-1}\| = \|T(x_{n-1}) - T(x_{n-2})\| \leqslant \alpha \|x_{n-1} - x_{n-2}\| .$$

Hence

$$\|x_{n+1} - x_n\| \leqslant \alpha \cdot \alpha \|x_{n-1} - x_{n-2}\| = \alpha^2 \|x_{n-1} - x_{n-2}\| .$$

Therefore by continuing this process we have

$$\|x_{n+1} - x_n\| \leqslant \alpha^n \|x_1 - x_0\| = \alpha^n M , \qquad \text{where } M = \|x_1 - x_0\| .$$

Using this inequality we will show that the sequence $\{x_n\}$ is a CAUCHY sequence:

$$\|x_{n+p} - x_n\| \leqslant \|x_{n+p} - x_{n+p-1}\| + \|x_{n+p} - x_{n+p-2}\| + \|x_{n-1} - x_n\|$$

$$\leqslant M \alpha^{n+p-1} + M \alpha^{n+p-2} + \ldots + M \alpha^n$$

$$\leqslant M \alpha^n (1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1}) \leqslant M \frac{\alpha^n}{1 - \alpha} .$$

Since $\alpha < 1$, $\frac{\alpha^n}{1-\alpha} M$ tends to zero as n tends to ∞, and hence $\{x_n\}$ is a CAUCHY sequence.

Since K is a closed subset of a finite dimensional BANACH space, K is complete. Hence x_n converges to some point x_0 in K. Let $x_0 = T \lim_{n \to \infty} x_n$. Then by the virtue of continuity of T, $Tx_0 = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = x_0$. Thus the existence of a fixed point is proved.

Uniqueness. Assume that x and y are two fixed points of T, i.e., $T(x) = x$ and $T(y) = y$, $x \neq y$. Then since T is contracting mapping we have

$$\|x - y\| = \|T(x) - T(y)\| \leqslant \alpha \|x - y\| ,$$

i.e., $(1 - \alpha) \|x - y\| = 0$, whence $\|x - y\| = 0$, so that x coincides with y. Theorem is proven.

One can remark that the theorem may be sharpened on replacing the condition that K is a closed and convex subset of a finite dimensional BANACH space by K as a closed subset of a BANACH space, or even in general by taking T as a contracting mapping of a BANACH space E into itself. But here our aim has been to prove the nonexpanding principle with the help of this theorem for which the above conditions are necessary. Since for the existence of a fixed point for nonexpansive mapping the following conditions on the space are necessary.

(1) The space should be closed, (2) bounded, (3) convex and (4) should be reflexive, as can be seen by following examples.

Example 1. Consider the HILBERT [3] space H ($H = R$ will suffice). Let D be the interior of the unit ball $C = \{x \mid \|x\| < 1\}$. The mapping T defined by

$$T(x) = x + a/2 ,$$

where a is any vector in H with unit norm is nonexpansive but it has no fixed point.

Example 2. A transformation in a BANACH space is an isometry which is a nonexpansive mapping but it has no fixed point.

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[3] For definition of HILBERT space see [17].

Example 3. Let H be a HILBERT space ($H = R$ will suffice again). Let D be a set containing two elements x and y. If we define T: $D \to D$ by $T(x) = y$ and $T(y) = x$. Then T is an isometry and it has no fixed point.

Example 4. The BANACH space $C[0, 1]$ of continuous functions is not reflexive. Let $K = \{T \in C[0,1]: T(0) = 0, T(1) = 1, 0 \leqslant T(x) < 1\}$, K is bounded, closed and convex. Define the mapping f by

$$f(T(x)) = x\,T(x)\,.$$

Clearly, f maps K into itself, T is nonexpansive, but T has no fixed point.

Before proving the results in BANACH spaces, we would like to state some fundamental properties without proofs. The proofs may be found in any standard book on Functional Analysis such as *Functional Analysis* by ALBERT WILANSKY.

1. A finite-dimensional normed space is reflexive.
2. A finite-dimensional normed space is complete.
3. Any finite-dimensional linear subspace of a normed space is closed.

8.5. – Theorem. *Let K be a closed and convex subset of a finite-dimensional Banach space and let T be a nonexpansive mapping of K into itself. Then T has a unique fixed point.*

Proof. Let $T_n = (1 - 1/n)\,T(x_n) + N$ (set of natural numbers). Then T_n is a contracting mapping and hence by Theorem **8.4** has a unique fixed point, say x_n, i.e., $T_n(x_n) = x_n = (1 - 1/n)\,T(x_n)$.

Since K is closed and bounded, therefore K is compact, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ which converges to a point x in K. Now

$$x_{n_i} = (1 - 1/n_i)T(x_{n_i})\,.$$

Taking the limit as $n \to \infty$ and using the fact that T is continuous, we get $x = T(x)$ and hence x is a unique fixed point.

9.1. – Theorem. *Let E be a Banach space, and let T_n $(n = 1, 2, \ldots)$ be contraction mappings of E into itself with the same constant $\alpha < 1$, and with fixed points u_n $(n = 1, 2, \ldots)$. Suppose that $\lim_{n \to \infty} T_n\,x = T_x$ for every $x \in E$, where T is a mapping from E into itself. Then T has a unique fixed point u and $\lim_{n \to \infty} u_n = u$.*

Proof. Since $\alpha < 1$ is the same constant for all n,

$$\lim_{n\to\infty} \|T_n x - T_n y\| \leqslant \alpha \|x - y\| .$$

But

$$\lim_{n\to\infty} \|T_n x - T_n y\| = \|\lim_{n\to\infty} (T_n x) - \lim_{n\to\infty} (T_n y)\| .$$

Because norm is continuous or $\|T(x) - T(y)\| \leqslant \alpha \|x - y\|$. As

$$\|\lim_{n\to\infty} (T_n x) - \lim_{n\to\infty} (T_n y)\| = \|T(x) - T(y)\| .$$

Thus T is a contraction mapping and has a unique fixed point by Theorem **8.4.**
We have, for each $n = 1, 2, \ldots,$

$$\|u_n - T_n^m x_0\| \leqslant \frac{\alpha^n}{1-\alpha} \| T_n x_0 - x_0\| , \qquad x_0 \in E .$$

Setting $m = 0$ and $x_0 = u$, we have

$$\|u_n - u\| \leqslant \frac{1}{1-\alpha} \|T_u - u\| = \frac{1}{1-\alpha} \|T_n u - T_u\| .$$

But $\|T_n u - T_u\| \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} \|u_n - u\| = 0 \qquad \Longrightarrow \qquad \lim_{n\to\infty} u_n = u .$$

The following corollary is a direct consequence of the above theorem.

Corollary. *Let E be a Banach space and let T_n $(n = 1, 2, \ldots)$ be contraction mappings of E into itself with constants α_n $(n = 1, 2, \ldots)$ such that $\alpha_{n+1} \leqslant \alpha_n$ for each n, and with fixed points u_n $(n = 1, 2, \ldots)$. Suppose that $\lim_{n\to\infty} (T_n x) = Tx$ for every $x \in E$, where T is a mapping from E into itself. Then T has a unique fixed point u and $\lim_{n\to\infty} u_n = u$.*

Proof. Since $\alpha_{n+1} \leqslant \alpha_n$ for all n, it follows that $\lim_{n\to\infty} \alpha_{n+1} < 1$. Now

$\lim_{n\to\infty} \|T_n x - T_n y\| \leqslant \alpha_n \|x-y\|$ since norm is continuous. Hence

$$\lim_{n\to\infty} \|T_n x - T_n y\| = \|\lim_{n\to\infty}(T_n x) - \lim_{n\to\infty}(T_n y)\| = \|Tx - Ty\| ,$$

$$\|Tx - Ty\| \leqslant \alpha_n \|x-y\| .$$

$Tx = \lim_{n\to\infty}(T_n x)$ is a contraction mapping. Moreover α_1 will be constant for T_n ($n = 1, 2, \ldots$). Thus the proof follows by Theorem **9.1** by replacing α by α_1.

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Abstract.

The contraction mapping principle of Banach remains the most fruitful for proving the existence theorems in Analysis. For this purpose the extensions of the theorem are of continuing interest. In the present paper generalization of a few results of Edelstein have been given. In section 4 *it has been shown that on a straight set of type* II *every* (ε, λ)*-uniformly local contraction is also a global contraction. In section* 5 *taking the set of continuous and contraction mappings on a metrizable subspace* P *of a metric space* X *into itself the necessary and sufficient condition for the compactness of* X *is shown. The remaining sections contain the generalizations of a result due to De Marr* [6] *and some results on eventual global contraction. In the last, two theorems in Banach spaces are added which generalize the result of Bers* [2].

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