Riv. Mat. Univ. Parma (2) **10** (1969), 181-184

B. D. MALVIYA (*)

Some Theorems on Operators. (**)

A bounded operator T defined on a HILBERT space H is said to be hyponormal if $\| T^{*}x\| \leq \| Tx\|$ for any $x \in H$. This definition has also an equivalent form: We say that the operator T is hyponormal if $T^{*} T \geq TT^{*}$. The notion of hyponormality was introduced by HALMOS [2].

A closed linear maniforld in the HILBERT space is called a subspace of H. The set of all complex numbers λ for which $(T - \lambda I)^{-1}$ does not exist is called the spectrum of T and is denoted by $\sigma(T)$.

An operator T defined on a HILBERT space H is said to be compact (completely continuous) if it maps every bounded set into a compact set. In other words an operator T is called completely continuous if there exists a sequence of elements $x_n \in H$ $(\|x_n\| = 1)$ such that $\{Tx_n\}$ contains a convergent subsequence. Recently another class of operators known as completely continuous operators have been in prominence. The first fundamental result concerning completely continuous hyponormal operators was proved by ANDÔ [1]. He proved *inter allia* that a completely continuous hyponormal operator is necessarily normal.

Here in this Note we prove certain results concerning hyponormal operators.

Theorem 1. *Let* T *be a hyponormal operator defined on a Hilbert space H and let M be a closed subspace of H such that*

$$M = \{x: \ Tx = \lambda x, \ x \neq \varnothing\},$$

then the subspace M will be spanned by eigen vectors of T.

(*) Indirizzo: Department of Mathematics, University of Allahabad, Allahabad, India.

(**) Ricevuto: 19-X-1967.

Proof. Let there be a vector Φ not orthogonal to M. We may express it as $\Phi = \Phi_1 + \Phi_2$, where $\Phi_1 = \varnothing \in M$ and $\Phi_2 \in M^{\perp}$. Hence $\lambda\Phi = \lambda\Phi_1 + \lambda\Phi_2 = T\Phi = T\Phi_1 + T\Phi_2$ and so $T\Phi_1 - \lambda\Phi_1 = \lambda\Phi_2 - T\Phi_2$. It is easily verified that M is invariant under T. Now, on account of invariant property of T, we have $T\Phi_1 \in M$ and $T\Phi_2 \in M^{\perp}$. Now only the null element is common between M and $M^{\perp}$. Hence $T\Phi_1 = \lambda\Phi_1$ where $\Phi_1 \neq \varnothing$. Hence the result follows from Theorem 2 (cf. [3]).

Theorem 2. *Let* T *be a hyponormal operator and* M *a set such that*

$$M = \{x: \quad \| Tx\| = \| T^*x\|,\ Tx = \lambda_i x,\ x \in H\},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the eigen values of* T. *Further let* $\sum_1^{\infty} \lambda_i^n \omega_i = 0$ *and* $\sum_{i=1}^{\infty} |\omega_i| < \infty$ *imply that* $\omega_1 = \omega_2 = \ldots = 0$. *Then* M *is generated by eigen vectors of* T.

Proof. Under the hypothesis of the Theorem, it is easy to see that M is a closed subspace invariant under T and T*. It is also known that the eigenspaces corresponding to different eigenvalues are orthogonal. So any $x \in M$ may be expressed as $x = \sum_{i=1}^{\infty} x_i$ where $x_i \in N_T(\lambda_i) = \{x_i\colon Tx_i = \lambda_i x_i\}$. Let $y \perp M$, we write $y = \sum_{i=1}^{\infty} \eta_i$ and $\eta_i \in N_T(\lambda_i)$. Since $T^n x \in M$, $n \geqslant 0$, we have

$$(T^n x,\ y) = \sum_{i=1}^{\infty} \lambda_i^n (\eta_i,\ x_i) = 0.$$

Putting $(\eta_i,\ x_i) = \omega_i$, we have $\sum_{i=j}^{\infty} \lambda_j^n \omega_i = 0$ where $n \geqslant 0$. Now since

$$\sum_{i=1}^{\infty} \| x_i\|^2 < \infty, \qquad \sum_{i=1}^{\infty} \| \eta_i\|^2 < \infty,$$

we further have $\sum_{i=1}^{\infty} |\omega_i| < \infty$. So it follows that $(y,\ x_i) = 0$ for $i = 1, 2, \ldots$. It means that every $x_i \in M$ and the conclusion of the theorem follows.

Theorem 3. *Let* T *be a hyponormal operator and let*

$$M_i = \{x: \quad x \in H,\ Tx = \lambda_i x\}.$$

where λ_1, λ_2, ... *are eigenvalues of* T. *Then* $\sigma(T) = \cup\, \sigma(T_{|M_i})$, *where* $T_{|M_i}$ *denotes the restriction of* T *to* M_i.

Proof. Now M_i's are invariant under T. The subspaces M_i, M_j are mutually orthogonal for $i \neq j$. The restriction of T to M_i, denoted by $T_{|M_i}$ is the mapping $T_{|M_i}$: $M_i \to M_i$ defined by $T_{|M_i} x = Tx$. Let $\lambda \in' \cup\, \sigma(T_{|M_i})$.
Then, supposing $T_\lambda = T - \lambda I$, we have

$$\| T_{\lambda|M_1} x_1 \| = \| Tx_1 \| \geqslant c \| x_1 \|, \qquad \| T_{\lambda|M_2} x_2 \| = \| Tx_2 \| \geqslant c \| x_2 \|$$

and so on. Any $x \in H$ may be expressed in the form $x = \sum_{i=1}^{\infty} x_i$ where $x_i \in M_i$ $(i = 1, 2, \ldots)$. Then

$$\| T_\lambda x \| = \| \sum_{i=1}^{\infty} T_{\lambda|M_i} x_i \| = \| \sum_{i=1}^{\infty} Tx_i \| \geqslant c\, (\sum_{i=1}^{\infty} \| x_i \|^2)^{1/2} = c \| x \|.$$

But this is indicative of the fact that $\lambda \in' \sigma(T)$. Hence the Theorem is proved.

It is known that if T is completely continuous then T* is also completely continuous. But the converse relationship is not known. In the theorem given below we have pointed out that under the hyponormality of T* and complete continuity of T*, T is completely continuous. We state this elementary result in the form of

Theorem 4. *If* T* *is completely continuous hyponormal operator, then* T *is completely continuous.*

Proof. By definition of hyponormality, we have

$$\| T(x_n - x_m) \| \leqslant \| T^*(x_n - x_m) \| \to 0 \qquad (m, n \to \infty),$$

where $\{x_n\}$ is any bounded sequence of elements $x_n \in H$, such that $\| x_n \| = 1$.

Theorem 5. *If* T *is a hyponormal operator and* λ *an isolated point in the spectrum of* T, *then* λ *belongs to the point spectrum of* T.

Proof. Since λ is an isolated point in the spectrum of T, we can find a circle with centre λ such that $|\mu - \lambda| = r$ is free from the points of $\sigma(T)$

except the point λ which is inside the circle. Then

$$P = \frac{1}{2\pi i} \int_{|\mu - \lambda| = r} (\mathrm{T} - \mu\,\mathrm{I})^{-1}\, d\mu$$

is a projection which commutes with T and

$$\|P\| \leqslant \frac{1}{2\pi} \int \|\,\mathrm{T} - \mu \mathrm{I}\,\|^{-1}\, d\mu \leqslant \frac{1}{2\pi}\, 2\pi\, r\, \frac{1}{r} = 1.$$

Thus P is self-adjoint. Now for $x \in PH$, we have

$$\|\,(\mathrm{T} - \lambda \mathrm{I})x\| = \left\| \frac{1}{2\pi i} \int (\mu - \lambda)(\mathrm{T} - \lambda \mathrm{I})^{-1} x\, d\mu \right\| \leqslant r\,.$$

Letting $r \to 0$, we have $\mathrm{T}x = \lambda x$. It proves our assertion. It is also proved by STAMPFLI [4].

References.

[1] T. ANDÔ, *On hyponormal operators*, Proc. Amer. Math. Soc. **14** (1963), 290-291.

[2] P. R. HALMOS, *Normal dilations and extensions of operators*, Summa Brasil. Math. **2** (1950), 124-134.

[3] B. D. MALVIYA, *On hyponormal operators*, Riv. Mat. Univ. Parma (2) **10** (1969), 9-12.

[4] J. G. STAMPFLI, *Hyponormal operators*, Pacific J. Math. **12** (1962), 1453-1458.

* * *