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Distribution of Binomial Coefficients. (**)

1. - Introduction.

Let p be a fixed prime and let $\theta_j(n)$ denote the number of binomial coefficients

$$\binom{n}{k} \qquad (0 \leqslant k \leqslant n)$$

divisible by exactly p^j. In a previous paper [1] the writer has obtained numerous properties of $\theta_j(n)$ and also the closely related function $\psi_j(n)$. The function $\psi_j(n)$ is defined as the number of pruducts

$$(n+1)\binom{n}{k} \qquad (0 \leqslant k \leqslant n)$$

divisible by exactly p^j. It was proved, for example, that $\theta_j(n)$ and $\psi_j(n)$ satisfy the following mixed recurrences:

$$(1.1) \quad \theta_j(a_0 + ap) = (a_0 + 1)\,\theta_j(a) + (p - a_0 - 1)\,\psi_{j-1}(a-1) \quad (0 \leqslant a_0 < p),$$

$$(1.2) \quad \psi_j(a_0 + ap) = (a_0 + 1)\,\theta_j(a) + (p - a_0 - 1)\,\psi_{j-1}(a-1) \quad (0 \leqslant a_0 < p-1),$$

$$(1.3) \quad \psi_j(p - 1 + ap) = p\,\psi_{j-1}(a),$$

where $j \geqslant 1$, a is an arbitrary nonnegative integer and $\psi_j(-1) = 0$ for all $j \geqslant 0$.

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The object of the present paper is to sharpen the results of [**1**]. In place of $\theta_j(n)$, $\psi_j(n)$ we consider the functions $F_j(n;\ x,\ y)$, $G_j(n;\ x,\ y)$ defined as follows:

$$F_j(n;\ x,\ y) = \sum_{\substack{a+b=n \\ E(a,\ b)=j}} x^a y^b ,$$

$$G_j(n;\ x,\ y) = \sum_{\substack{a+b=n \\ E'(a,\ b)=j}} x^a y^b ,$$

where $E(a,\ b)$ denotes the largest value of k such that

$$p^k \,\Big|\, \binom{a+b}{b}$$

and $E'(a,\ b)$ denotes the largest value of k such that

$$p^k \,\Big|\, (a+b+1) \binom{a+b}{b} .$$

Clearly

$$F_j(n;\ x,\ x) = x^n\, \theta_j(n), \qquad G_j(n;\ x,\ x) = x^n\, \psi_j(n) .$$

We shall show that most of the results concerning $\theta_j(n)$, $\psi_j(n)$ obtained in [**1**] can be extended to the functions $F_j(n; x, y)$, $G_j(n; x, y)$. In particular, the recurrences (1.1), (1.2), (1.3) become

$$(1.4) \qquad F_j(a_0 + ap;\ x,\ y) = c_{a_0}(x,\ y)\, F_j(a;\ x^p,\ y^p) + \\ + (xy)^{a_0+1}\, c_{p-a_0-2}(x,\ y)\, G_{j-1}(a-1;\ x^p,\ y^p) \quad (0 \leqslant a < p),$$

$$(1.5) \qquad G_j(a_0 + ap;\ x,\ y) = c_{a_0}(x,\ y)\, F_j(a;\ x^p,\ y^p) + \\ + (xy)^{a_0+1}\, c_{p-a_0-2}(x,\ y)\, G_{j-1}(a-1;\ x^p,\ y^p) \quad (0 \leqslant a < p-1),$$

$$(1.6) \qquad G_j(p-1 + ap;\ x,\ y) = c_{p-1}(x,\ y)\, G_{j-1}(a;\ x^p,\ y^p),$$

where

$$c_a(x,\ y) = \frac{x^{a+1} - y^{a+1}}{x - y} .$$

More generally we obtain recurrences for the functions

$$F_j(a + bp^s;\ x,\ y), \qquad G_j(a + bp^s;\ x,\ y);$$

these are contained in (5.10), (5.11), (5.12) below. As an application we get

$$(1.7) \qquad (xy)^{p^s-1}\, F_{s-t}\left(p^s - a - 1\,;\, \frac{1}{x}\,,\, \frac{1}{y}\right) =$$

$$= G_t(a + p^s;\ x,\ y) - (x^{p^s} + y^{p^s})\, G_t(a;\ x,\ y) \qquad (0 \leqslant a < p^s - 1)\ .$$

$$(1.8) \qquad (xy)^{p^s-1}\, G_{s-t}\left((p^s - a - 1\,;\, \frac{1}{x}\,,\, \frac{1}{y}\right) =$$

$$= F_t(a + p^s;\ x,\ y) - (x^{p^s} + y^{p^s})\, F_t(a;\ x,\ y) \qquad (0 \leqslant a < p^s - 1)\ ,$$

where $1 \leqslant t < s$.

In [1] explicit formulas were obtained for

$$\theta_j(an_r), \qquad \theta_j(an_r - 1), \qquad (0 < a < p)\ ,$$

where $n_r = (p^r - 1)/(p - 1)$. Also explicit results were obtained for

$$S_j(r) = \sum_{a=0}^{p^r-1} \theta_j(a)\ , \qquad S'_j(r) = \sum_{a=0}^{p^r-1} \psi_j(a)\ .$$

We have not been able to get satisfactory generalizations of these results. For partial generalizations see §§ **4**, **6** below.

2. - Preliminaries.

For $a,\ b \geqslant 0$, let $E(a,\ b)$ denote the largest value of k such that

$$(2.1) \qquad p^k \left| \binom{a+b}{a} \right. ;$$

let $E'(a,\ b)$ denote the largest value of k such that

$$(2.2) \qquad p^k \left| \, (a + b + 1) \binom{a+b}{a} \right. .$$

Let $\theta_j(n)$ denote the number of solutions of

$$E(a,\ n-a) = j \qquad (0 \leqslant a \leqslant n); \tag{2.3}$$

let $\psi_j(n)$ denote the number of solutions of

$$E'(a,\ n-a) = j \qquad (0 \leqslant a \leqslant n)\,. \tag{2.4}$$

Put

$$n = c_0 + c_1 p + \ldots + c_r p^r \qquad (0 \leqslant c_k < p)\,,$$

$$S(n) = c_0 + c_1 + \ldots + c_r\,,$$

where p is a fixed prime. If $p^{\nu(n)}$ denotes the highest power of p that divides $n!$, it is well known that

$$\nu(n) = \frac{n - S(n)}{p-1}\,. \tag{2.5}$$

It follows that

$$E(a,\ b) = \frac{S(a) + S(b) - S(a+b)}{p-1}\,, \tag{2.6}$$

$$E'(a,\ b) = \frac{S(a) + S(b) - S(a+b+1) + 1}{p-1}\,. \tag{2.7}$$

We now define

$$F(a,\ y,\ z) = \sum_{i=0}^{\infty} z^j F_j(x,\ y)\,, \qquad F_j(x,\ y) = \sum_{E(a,b)=j} x^a y^b\,, \tag{2.8}$$

$$G(x,\ y,\ z) = \sum_{j=0}^{\infty} z^j G_j(x,\ y)\,, \qquad G_j(x,\ y) = \sum_{E'(a,\,b)=j} x^a y^b\,. \tag{2.9}$$

We also define

$$F_j(n;\ x,\ y) = \sum_{\substack{a+b=n \\ E(a,\,b)=j}} x^a y^b\,, \tag{2.10}$$

$$G_j(n;\ x,\ y) = \sum_{\substack{a+b=n \\ E'(a,\,b)=1}} x^a y^b\,, \tag{2.11}$$

so that

$$F_j(x,\ y) \quad = \sum_{n=0}^{\infty} F_j(n;\ x,\ y) \tag{2.12}$$

and

$$G_j(x,\ y) \quad = \sum_{n=0}^{\infty} G_j(n;\ x,\ y)\ . \tag{2.13}$$

It is also evident that

$$F(x,\ y,\ z) \quad = \sum_{a,\, b=0}^{\infty} x^a\, y^b\, z^{E(a,\, b)} \tag{2.14}$$

and

$$G(x,\ y,\ z) \quad = \sum_{a,\, b=0}^{\infty} x^a\, y^b\, z^{E'(a,\, b)}\ . \tag{2.15}$$

The following notation will be useful. Put

$$c_r(x,\ y) \quad = \quad \sum_{s=0}^{r} x^s\, y^{r-s} = \frac{x^{r+1} - y^{r+1}}{x - y} \qquad (0 \leqslant r < p)\ , \tag{2.16}$$

$$f_0(x,\ y) \quad = \sum_{a+b \leq p-1} x^a\, y^b \quad = \sum_{r=0}^{p-1} c_r(x,\ y)\ , \tag{2.17}$$

$$g_0(x,\ y) \quad = \sum_{\substack{a,\ b \leq p-1 \\ a+b \geq p}} x^a\, y^b = \sum_{r=1}^{p-1} (xy)^r\, c_{p-r-1}(x,\ y)\ , \tag{2.18}$$

$$f_1(x,\ y) \quad = \sum_{a+b < p-1} x^a\, y^b = \sum_{r=0}^{p-2} c_r(x,\ y)\ , \tag{2.19}$$

$$g_1(x,\ y) \quad = \sum_{\substack{a,\ b \leq p-1 \\ a+b \geq p-1}} x^a\, y^b = \sum_{r=0}^{p-1} (xy)^r\, c_{p-r-1}(x,\ y)\ . \tag{2.20}$$

We recall that if

$$a = a_0 + a_1 p + a_2 p^2 + \dots \qquad (0 \leqslant a_i < p),$$

$$b = b_0 + b_1 p + b_2 p^2 + \dots \qquad (0 \leqslant b_i < p),$$

then $\binom{a+b}{a}$ is prime to p if and only if

$$a_i + b_i < p \qquad (i = 0, 1, 2, \dots).$$

Thus, by (2.8),

$$F_0(x, y) = \sum_{E(a, b)=0} x^a y^b$$

$$= \sum_{a_0+b_0<p} x^{a_0} y^{b_0} \sum_{a_1+b_1<p} x^{a_1 p} y^{b_1 p} \dots .$$

Therefore, by (2.17),

$$F_0(x, y) = \prod_{k=0}^{\infty} f_0(x^{p^k}, y^{p^k}). \tag{2.21}$$

Similarly, by (2.9) and (2.19),

$$G_0(x, y) = f_1(x, y) \prod_{k=1}^{\infty} f_0(x^{p^k}, y^{p^k}). \tag{2.22}$$

3. - Recurrences.

It follows from (2.6), (2.7) and (2.14) that

$$F(x, y, z) = \sum_{a, b=0}^{\infty} x^a y^b z^{(S(a)+S(b)-S(a+b))/(p-1)}$$

$$= \sum_{a_0, b_0=0}^{p-1} \sum_{a, b=0}^{\infty} x^{a_0+ap} y^{b_0+bp} \cdot z^{(S(a_0+ap)+S(b_0+bp)-S(a_0+b_0+ap+bp))/(p-1)} .$$

Since, for $0 \leqslant a_0 < p$, $0 \leqslant b_0 < p$,

$$S(a_0 + ap) = a_0 + S(a), \qquad S(b_0 + bp) = b_0 + S(b),$$

while

$$S(a_0 + b_0 + ap + bp) = \begin{cases} a_0 + b_0 + S(a + b) & (a_0+b_0 < p) \\ a_0 + b_0 - p + S(a + b + 1) & (a_0+b_0 \geqslant p), \end{cases}$$

it follows that

$$F(x, y, z) = \sum_{a_0, b_0 < p} \sum_{a, b=0}^{\infty} x^{a_0+ap} y^{b_0+bp} z^{\{S(a)+S(b)-S(a+b)\}/(p-1)} +$$

$$+ \sum_{\substack{a_0, b_0 < p \\ a_0+b_0 \geq p}} \sum_{a, b=0}^{\infty} x^{a_0+ap} y^{b_0+bp} z^{\{p+S(a)+S(b)-S(a+b+1)\}/(p-1)}$$

$$= \sum_{a_0+b_0<p} x^{a_0} y^{b_0} \sum_{a,b=0}^{\infty} x^{ap} y^{bp} z^{E(a,b)} + z \sum_{\substack{a_0, b_0 < p \\ a_0+b_0 \geq p}} x^{a_0} y^{b_0} \sum_{a,b=0}^{\infty} x^{ap} y^{bp} z^{E'(a,b)}.$$

In view of (2.17) and (2.18), this reduces to

$$F(x, y, z) = f_0(x, y) F(x^p, y^p, z) + z g_0(x, y) G(x^p, y^p, z). \tag{3.1}$$

In the next place

$$G(x, y, z) = \sum_{a, b=0}^{\infty} x^a y^b z^{(S(a)+S(b)-S(a+b+1)+1)/(p-1)}$$

$$= \sum_{a_0, b_0=0}^{p-1} \sum_{a, b=0}^{\infty} x^{a_0+ap} y^{b_0+bp} z^{\{S(a_0+ap)+S(b_0+bp)-S(a_0+b_0+1+ap+bp)-1\}/(p-1)}$$

$$= \sum_{a_0+b_0<p-1} \sum_{a,b=0}^{\infty} x^{a_0+ap} y^{b_0+bp} z^{\{S(a)+S(b)-S(a+b)\}/(p-1)} +$$

$$+ \sum_{\substack{a_0, b_0 \leq p-1 \\ a_0+b_0 \geq p-1}} \sum_{a, b=0}^{\infty} x^{a_0+ap} y^{b_0+bz} z^{\{p+S(a)+S(b)-S(a+b+1)\}/(p-1)}$$

$$= \sum_{a_0+b_0<p-1} x^{a_0} y^{b_0} \sum_{a, b=0}^{\infty} x^{ap} y^{bp} z^{E(a, b)} + z \sum_{\substack{a_0, b_0 \leq p-1 \\ a_0+b_0 \geq p-1}} x^{a_0} y^{b_0} \sum_{a, b=0}^{\infty} x^{ap} y^{bp} z^{E'(a,b)}.$$

Hence, by (2.19) and (2.20) we get

$$G(x,\ y,\ z) = f_1(x,\ y)\, F(x^p,\ y^p,\ z) + z\, g_1(x,\ y)\, G(x^p,\ y^p,\ z)\,. \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$\begin{bmatrix} F(x,\ y,\ z) \\ G(x,\ y,\ z) \end{bmatrix} = \begin{bmatrix} f_0(x,\ y) & z\, g_0(x,\ y) \\ f_1(x,\ y) & z\, g_1(x,\ y) \end{bmatrix} \begin{bmatrix} F(x^p,\ y^p,\ z) \\ G(x^p,\ y^p,\ z) \end{bmatrix}, \tag{3.3}$$

where

$$\begin{bmatrix} F(x,\ y,\ z) \\ G(x,\ y,\ z) \end{bmatrix}, \qquad \begin{bmatrix} F(x^p,\ y^p,\ z) \\ G(x^p,\ y^p,\ z) \end{bmatrix}$$

denote column vectors. Since

$$F(0,\ 0,\ z) = G(0,\ 0,\ z) = 1,$$

it is evident that (3.3) implies

$$\begin{bmatrix} F(x,\ y,\ z) \\ G(x,\ y,\ z) \end{bmatrix} = \prod_{k=0}^{\infty} \begin{bmatrix} f_0(x^{p^k},\ y^{p^k}) & z\, g_0(x^{p^k},\ y^{p^k}) \\ f_1(x^{p^k},\ y^{p^k}) & z\, g_1(x^{p^k},\ y^{p^k}) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.4}$$

The order of the factors on the right of (3.4) must not be altered.

Comparing coefficients of z^j on both sides of (3.1), we get

$$F_j(x,\ y) = f_0(x,\ y)\, F_j(x^p,\ y^p) + g_0(x,\ y)\, G_{j-1}(x^p,\ y^p)\,. \tag{3.5}$$

Similarly, by (3.2),

$$G_j(x,\ y) = f_1(x,\ y)\, F_j(x^p,\ y^p) + g_1(x,\ y)\, G_{j-1}(x^p,\ y^p)\,. \tag{3.6}$$

We now make use of (2.12) and (2.13) and express $f_0(x,\ y),\ \dots,\ g_1(x,\ y)$ in terms of $c_r(x,\ y)$ by means of (2.17), ..., (2.20). It then follows from (3.5) and (3.6) that

$$\begin{aligned} F_j(a_0 + ap;\ x,\ y) &= c_{a_0}(x,\ y)\, F_j(a;\ x^p,\ y^p) + \\ &\quad + (xy)^{a_0+1}\, c_{p-a_0-2}(x,\ y)\, G_{j-1}(a-1;\ x^p,\ y^p) \qquad (0 \leqslant a < p), \end{aligned} \tag{3.7}$$

$$\begin{aligned} G_j(a_0 + ap;\ x,\ y) &= c_{a_0}(x,\ y)\, F_j(a;\ x^p,\ y^p) + \\ &\quad + (xy)^{a_0-1}\, c_{p-a_0-2}(x,\ y)\, G_{j-1}(a-1;\ x^p,\ y^p) \qquad (0 \leqslant a < p-1), \end{aligned} \tag{3.8}$$

(3.9) $$G_j(p-1+ap;\ x,\ y) = c_{p-1}(x,\ y)\ G_{j-1}(a;\ x^p,\ y^p)\ .$$

In (3.7), (3.8), (3.9) it is assumed that $j>0$, $a \geqslant 0$; also it is understood that

$$G_j(-1) = 0 \qquad (j = 0,\ 1,\ \ldots)\ .$$

If we define $\psi_{-1}(a) = 0$ then the formulas hold for $j=0$ also.

It follows from (3.7) and (3.8) that

(3.10) $$G_j(a;\ x,\ y) = F_j(a;\ x,\ y) \qquad (p \nmid a+1)$$

as is also evident from the definition. On the other hand, if $a_0 = p-1$, (3.7) becomes

(3.11) $$F_j(p-1+ap;\ x,\ y) = c_{p-1}(x,\ y)\ F_j(a;\ x^p,\ y^p)\ .$$

This evidently implies

(3.12) $$F_j(ap^s-1;\ x,\ y) = \prod_{k=0}^{s-1} c_{p-1}(x^{p^k},\ y^{p^k})\cdot F_j(a-1;\ x^{p^s},\ y^{p^s}) \qquad (a \geqslant 1)\ .$$

Since, by (2.16),

$$c_{p-1}(x,\ y) = \frac{x^p-y^p}{x-y}\ ,$$

(3.12) reduces to

(3.13) $$F_j(ap^s-1;\ x,\ y) = \frac{x^{p^s}-y^{p^s}}{x-y}\ F_j(a-1;\ x^{p^s},\ y^{p^s}) \qquad (a \geqslant 1)\ .$$

On the other hand, by (3.9),

(3.14) $$G_j(ap^s-1;\ x,\ y) = \frac{x^{p^s}-y^{p^s}}{x-y}\ G_{j-s}(a-1;\ x^{p^s},\ y^{p^s}) \qquad (a \geqslant 1;\ j \geqslant s).$$

In view of (3.10), this gives

(3.15) $$G_j(ap^s-1;\ x,\ y) = \frac{x^{p^s}-y^{p^s}}{x-y}\ F_{j-s}(a-1;\ x^{p^s},\ y^{p^s}) \qquad (p \nmid a,\ j \geqslant s).$$

Next it is clear from (3.7) that

$$F_j(ap;\ x,\ y) = F_j(a;\ x^p,\ y^p) + x\,y\ c_{p-2}(x,\ y)\ G_{j-1}(a-1;\ x^p,\ y^p)\,.$$

Thus

$$F_j(ap^2;\ x,\ y) = F_j(ap;\ x^p,\ y^p) + x\,y\ c_{p-2}(x,\ y)\ G_{j-1}(ap-1;\ x^p,\ y^p)$$

$$= F_j(a;\ \ x^{p^2},\ y^{p^2}) + (xy)^p\ c_{p-2}(x^p,\ y^p)\ G_{j-1}(a-1;\ x^{p^2},\ y^{p^2}) +$$

$$+ x\,y\ c_{p-2}(x,\ y)\,\frac{x^{p^2}-y^{p^2}}{x^p-y^p}\,G_{j-2}(a-1;\ x^{p^2},\ y^{p^2})\,.$$

Iteration leads to

(3.16) $$F_j(ap^s;\ x,\ y) = F_j(a;\ x^{p^s},\ y^{p^s}) +$$

$$+ \sum_{t=1}^{s} (xy)^{p^{s-t}}\ c_{p-2}(x^{p^{s-t}},\ y^{p^{s-t}})\ \frac{x^{p^s}-y^{p^s}}{x^{p^{s-t+1}}-y^{p^{s-t+1}}}\cdot G_{j-t}(a-1;\ x^{p^s},\ y^{p^s})\,.$$

In particular, by (3.10), we have

(3.17) $$F_j(ap^s;\ x,\ y) = F_j(a;\ x^{p^s},\ y^{p^s}) +$$

$$+ \sum_{t=1}^{s} (xy)^{p^{s-t}}\ c_{p-2}(x^{p^{s-t}},\ y^{p^{s-t}})\ \frac{x^{p^s}-y^{p^s}}{x^{p^{s-t+1}}-y^{p^{s-t+1}}}\cdot F_{j-t}(a-1;\ x^{p^s},\ y^{p^s})\ \ (p+a)\,.$$

Since, for $0 < a < p$,

$$F_j(a;\ x,\ y) = \begin{cases} c_a(x,\ y) & (j=0) \\ 0 & (j>0)\,, \end{cases}$$

it can be verified that (3.17) implies

(3.18) $$F_j(ap^s;\ x,\ y) =$$

$$= \begin{cases} c_a(x^{p^s},\ y^{p^s}) & (j=0) \\ (xy)^{p^{s-t}}\ c_{p-2}(x^{p^{s-j}},\ y^{p^{s-j}})\ c_{a-1}(x^{p^s},\ y^{p^s})\,\dfrac{x^{p^s}-y^{p^s}}{x^{p^{s-j+1}}-y^{p^{s-j+1}}} & (0<j\leqslant s)\,. \end{cases}$$

As noted in [1]

$$\theta_0(n) = n + 1 \tag{3.19}$$

if and only if

$$n = ap^s + p^s - 1 \qquad (0 \leqslant a < p; \quad s \geqslant 0)\,. \tag{3.20}$$

Now, by (3.13), for $0 \leqslant a < p$,

$$F_j(ap^s + p^s - 1;\ x,\ y) = \frac{x^{p^s} - y^{p^s}}{x - y} F_j(a;\ x^{p^s},\ y^{p^s})$$

$$= \begin{cases} \dfrac{x^{p^s} - y^{p^s}}{x - y}\, c_a(x^{p^s},\ y^{p^s}) & (j = 0) \\ 0 & (j > 0) \end{cases}$$

in agreement with (3.19).

4. - Some special evaluations.

Put

$$n_r = \frac{p^r - 1}{p - 1} = 1 + p\, n_{r-1}\,. \tag{4.1}$$

Then, by (3.7) and (3.10), for $0 < a < p$,

$$\begin{aligned} F_j(an_{r+1};\ x,\ y) &= c_a(x,\ y)\, F_j(an_r;\ x^p,\ y^p) \\ &\quad + (xy)^{a+1}\, c_{p-a-2}(x,\ y)\, F_{j-1}(an_r - 1;\ x^p,\ y^p)\,, \end{aligned} \tag{4.2}$$

while

$$\begin{aligned} F_j(an_{r+1} - 1;\ x,\ y) &= c_{a-1}(x,\ y)\, F_j(an_r;\ x^p,\ y^p) + \\ &\quad + (xy)^a\, c_{p-a-1}(x,\ y)\, F_{j-1}(an_r - 1;\ x^p,\ y^p)\,. \end{aligned} \tag{4.3}$$

Now put

$$\Theta(x, y; u, v) = \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} F_j(an_r; x, y)\, u^r v^j,$$

$$\bar{\Theta}(x, y; u, v) = \sum_{r=1}^{\infty} \sum_{j=0}^{\infty} F_j(an_r - 1; x, y)\, u^r v^j.$$

Then, by (4.3), for $0 < a < p$,

$$\bar{\Theta}(x, y; u, v) = u \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \{c_{a-1}(x, y)\, F_j(an_r; x^p, y^p) + $$
$$+ (xy)^a\, c_{p-a-1}(x, y)\, F_{j-1}(an_r - 1; x^p, y^p)\}\, u^r v^j,$$

so that

(4.4) $$\bar{\Theta}(x, y; u, v) =$$
$$= u\, c_{a-1}(x, y)\, \Theta(x^p, y^p; u, v) + u\, v\, (xy)^a\, c_{p-a-1}(x, y)\, \bar{\Theta}(x^p, y^p; u, v).$$

Similarly, by (4.2),

$$\Theta(x, y; u, v) = 1 + u \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \{c_a(x, y)\, F_j(an_r; x^p, y^p) + $$
$$+ (xy)^{a+1}\, c_{p-a-2}(x, y)\, F_{j-1}(an_r - 1; x^p, y^p)\}\, u^p v^j,$$

which gives

(4.5) $$\Theta(x, y; u, v) =$$
$$= 1 + u\, c_a(x, y)\, \Theta(x^p, y^p; u, v) + u\, v\, (xy)^{a+1}\, c_{p-a-2}(x, y)\, \bar{\Theta}(x^p, y^p; u, v).$$

The recurrences (4.4), (4.5) take on a somewhat simpler appearance if we make use of an operator E defined by

(4.6) $$E\, f(x, y) = f(x^p, y^p).$$

Then (4.5) becomes

(4.7) $$[1 - u\, c_a(x, y) E]\, \Theta(x, y; u, v) =$$
$$= 1 + u\, v\, (xy)^{a+1}\, c_{p-a-2}(x, y)\, E\, \bar{\Theta}(x, y; u, v),$$

while (4.4) becomes

$$(4.8)\quad [1 - u\,v\,(xy)^{a+1}\,c_{p-a-1}(x,\,y)\,E]\,\bar{\Theta}(x,\,y;\,u,\,v) = u\,c_{a-1}(x,\,y)\,E\,\Theta(x,\,y;\,u,\,v).$$

5. - More general recurrences.

It follows from (3.2) that

$$(5.1)\qquad \begin{bmatrix} F(x,\,y,\,z) \\ G(x,\,y,\,z) \end{bmatrix} = \prod_{k=0}^{s-1} \begin{bmatrix} f_0(x^{p^k},\,y^{p^k}) & z\,g_0(x^{p^k},\,y^{p^k}) \\ f_1(x^{p^k},\,y^{p^k}) & z\,g_1(x^{p^k},\,y^{p^k}) \end{bmatrix} \begin{bmatrix} F(x^{p^s},\,y^{p^s},\,z) \\ G(x^{p^s},\,y^{p^s},\,z) \end{bmatrix},$$

where s is an arbitrary positive integer. Put

$$(5.2)\qquad \prod_{k=0}^{s-1} \begin{bmatrix} f_0(x^{p^k},\,y^{p^k}) & z\,g_0(x^{p^k},\,y^{p^k}) \\ f_1(x^{p^k},\,y^{p^k}) & z\,g_1(x^{p^k},\,y^{p^k}) \end{bmatrix} = \begin{bmatrix} A_s(x,\,y,\,z) & B_s(x,\,y,\,z) \\ C_s(x,\,y,\,z) & D_s(x,\,y,\,z) \end{bmatrix}.$$

Then it is easily seen that

$$A_s(x,\,y,\,z) = \sum_{t=0}^{s-1} z^t \sum_{a=0}^{p^s-1} A_{st}(a;\,x,\,y)\,,$$

$$B_s(x,\,y,\,z) = \sum_{t=1}^{s} z^t \sum_{a=0}^{p^s-2} B_{st}(a+p^s;\,x,\,y)\,,$$

$$C_s(x,\,y,\,z) = \sum_{t=0}^{s-1} z^t \sum_{a=0}^{p^s-2} C_{st}(a;\,x,\,y)\,,$$

$$D_s(x,\,y,\,z) = \sum_{t=1}^{s} z^t \sum_{a=0}^{p^s-1} D_{st}(a+p^s-1;\,x,\,y),$$

where $A_{st}(a;\,x,\,y)$, ..., $D_{st}(a;\,x,\,y)$ are homogeneous of weight a in x, y.

Substituting in (5.1) and making use of (2.10) and (2.11), we get

$$(5.3)\qquad F_j(a+bp^s;\,x,\,y) = \sum_{t=0}^{s-1} A_{st}(a;\,x,\,y)\,F_{j-t}(b;\,x^{p^s},\,y^{p^s}) +$$

$$+ \sum_{t=1}^{s} B_{st}(a+p^s;\,x,\,y)\,G_{j-t}(b-1;\,x^{p^s},\,y^{p^s})\qquad (0 \leqslant a < p^s)\,,$$

$$(5.4)\qquad G_j(a + bp^s;\ x,\ y) = \sum_{t=0}^{s-1} C_{st}(a;\ x,\ y)\, F_{j-t}(b;\ x^{p^s},\ y^{p^s})$$

$$+ \sum_{t=1}^{s} D_{st}(a + p^s;\ x,\ y)\, G_{j-t}(b - 1;\ x^{p^s},\ y^{p^s}) \qquad (0 \leqslant a < p^s - 1)\,,$$

$$(5.5)\qquad G_j(p^s - 1 + bp^s;\ x,\ y) = \sum_{t=1}^{s} D_{st}(p^s - 1;\ x,\ y)\, G_{j-t}(b;\ x^{p^s},\ y^{p^s})\,.$$

In (5.3), (5.4), (5.5), b is an arbitrary nonnegative integer.

To determine $A_{st}(a;\ x,\ y),\ \ldots,\ D_{st}(a;\ x,\ y)$ we take $b = 0,\ 1$. For $b = 0$ we get

$$(5.6)\quad A_{st}(a;\ x,\ y) = F_t(a;\ x,\ y), \qquad C_{st}(a;\ x,\ y) = G_t(a;\ x,\ y), \qquad (0 \leqslant a < p^s)\,,$$

$$(5.7)\qquad D_{st}(p^s - 1;\ x,\ y) = G_t(p^s - 1;\ x,\ y) = \begin{cases} \dfrac{x^{p^s} - y^{p^s}}{x - y} & (t = s) \\ 0 & (t < s)\,. \end{cases}$$

For $b = 1$ we get

$$(5.8)\qquad B_{st}(a + p^s;\ x,\ y) = F_t(a + p^s;\ x,\ y) - (x^{p^s} + y^{p^s})\, F_t(a;\ x,\ y)$$

$$(0 \leqslant a < p^s)\,,$$

$$(5.9)\qquad D_{st}(a + p^s;\ x,\ y) = G_t(a + p^s;\ x,\ y) - (x^{p^s} + y^{p^s})\, G_t(a;\ x,\ y)$$

$$(0 \leqslant a < p^s - 1)\,.$$

We therefore get the following recurrences:

$$(5.10)\qquad F_j(a + bp^s;\ x,\ y) = \sum_{t=0}^{s-1} F_t(a;\ x,\ y)\, F_{j-t}(b;\ x^{p^s},\ y^{p^s}) +$$

$$+ \sum_{t=1}^{s} [F_t(a + p^s;\ x,\ y) - (x^{p^s} + y^{p^s})\, F_t(a;\ x,\ y)] \cdot G_{j-t}(b - 1;\ x^{p^s},\ y^{p^s})$$

$$(0 \leqslant a < p^s)\,,$$

$$(5.11)\qquad G_j(a+bp^s;\ x,\ y) = \sum_{t=0}^{s-1} G_t(a;\ x,\ y)\, F_{j-t}(b;\ x^{p^s},\ y^{p^s}) +$$

$$+ \sum_{t=1}^{s} \left[G_t(a+p^s;\ x,\ y) - (x^{p^s}+y^{p^s})\, G_t(a;\ x,\ y)\right] \cdot G_{j-t}(b-1;\ x,\ y)$$

$$(0 \leqslant a < p^s - 1),$$

$$(5.12)\qquad G_j(p^s-1+bp^s;\ x,\ y) = \frac{x^{p^s}-y^{p^s}}{x-y}\, G_{j-s}(b;\ x^{p^s},\ y^{p^s}).$$

For $s=1$, (5.10), (5.11), (5.12) reduce to (3.7), (3.8), (3.9), respectively.

In the next place, it is clear from (2.17), ..., (2.20) that

$$f_0(x,\ y) = (xy)^{p-1}\, g_1\!\left(\frac{1}{x},\ \frac{1}{y}\right), \qquad g_0(x,\ y) = (xy)^{p-1}\, f_1\!\left(\frac{1}{x},\ \frac{1}{y}\right).$$

It follows that

$$(xy)^{p-1}\, z \begin{bmatrix} f_0\!\left(\frac{1}{x},\ \frac{1}{y}\right) & \frac{1}{z}\, g_0\!\left(\frac{1}{x},\ \frac{1}{y}\right) \\ f_1\!\left(\frac{1}{x},\ \frac{1}{y}\right) & \frac{1}{z}\, g_1\!\left(\frac{1}{x},\ \frac{1}{y}\right) \end{bmatrix} = \begin{bmatrix} z\, g_1(x,\ y) & f_1(x,\ y) \\ z\, g_0(x,\ y) & f_0(x,\ y) \end{bmatrix},$$

$$(xy)^{p^s-1}\, z^s \prod_{k=0}^{s-1} \begin{bmatrix} f_0(x^{-p^k},\ y^{-p^k}) & \frac{1}{z}\, g_0(x^{-p^k},\ y^{-p^k}) \\ f_1(x^{-p^k},\ y^{-p^k}) & \frac{1}{z}\, g_1(x^{-p^k},\ y^{-p^k}) \end{bmatrix} =$$

$$= \prod_{k=0}^{s-1} \begin{bmatrix} z\, g_1(x^{p^k},\ y^{p^k}) & f_1(x^{p^k},\ y^{p^k}) \\ z\, g_0(x^{p^k},\ y^{p^k}) & f_0(x^{p^k},\ y^{p^k}) \end{bmatrix} = \begin{bmatrix} D_s(x,\ y,\ z) & C_s(x,\ y,\ z) \\ B_s(x,\ y,\ z) & A_s(x,\ y,\ z) \end{bmatrix}.$$

This gives

$$(xy)^{p^s-1} \sum_{t=0}^{s-1} z^{s-t} \sum_{a=0}^{p^s-1} A_{st}\!\left(a;\ \frac{1}{x},\ \frac{1}{y}\right) = \sum_{t=1}^{s} y^t \sum_{a=0}^{p^s-1} D_{st}(a+p^s-1;\ x,\ y),$$

$$(xy)^{p^s-1} \sum_{t=1}^{s} z^{s-t} \sum_{a=0}^{p^s-2} B_{st}\!\left(a+p^s;\ \frac{1}{x},\ \frac{1}{y}\right) = \sum_{t=0}^{s-1} y^t \sum_{a=0}^{p^s-2} C_{st}(a;\ x,\ y),$$

so that

$$D_{st}(2p^s - a - 2;\ x,\ y) = (xy)^{p^s-1} A_{s,\,s-t}\left(a;\ \frac{1}{x},\ \frac{1}{y}\right) \qquad (0 \leqslant a < p^s),$$

$$B_{st}(ap^s - a - 2;\ x,\ y) = (xy)^{p^s-1} C_{s,\,s-t}\left(a;\ \frac{1}{x},\ \frac{1}{y}\right) \qquad (0 \leqslant a < p^s - 1).$$

Comparison with (5.6), (5.7), (5.8) and (5.9) now gives

$$(5.13) \qquad (xy)^{p^s-1} F_{s-t}\left(a;\ \frac{1}{x},\ \frac{1}{y}\right) = G_t(2p^s - a - 2;\ x,\ y) - (x^{p^s} + y^{p^s})\ G_t(p^s - a - 2;\ x,\ y)$$

$$(0 \leqslant a < p^s - 1),$$

$$(5.14) \qquad (xy)^{p^s-1} G_{s-t}\left(a;\ \frac{1}{x},\ \frac{1}{y}\right) = F_t(2p^s - a - 2;\ x,\ y) - (x^{p^s} + y^{p^s})\ F_t(p^s - a - 2;\ x,\ y)$$

$$(0 \leqslant a < p^s - 1),$$

where $1 \leqslant t \leqslant s$.

It follows from either (5.13) or (5.14) that

$$\sum_{\substack{a,\,b=0 \\ E(a,b)=s-t}}^{p^s-1} x^a y^b = \sum_{\substack{a,b=0 \\ E'(a,b)=t}}^{p^s-1} x^{p^s-a-1}\ y^{p^s-b-1},$$

which is itself equivalent to

$$(5.15) \qquad E(a,\ b) + E'(p^s - a - 1,\ p^s - b - 1) = s \qquad (0 \leqslant a < p^s;\ 0 \leqslant b < p^s).$$

It is not difficult to give a direct proof of (5.15) using (2.6) and (2.7). Indeed one can easily prove the following stronger result.

Let

$$0 < u < p, \qquad 0 < v < p, \qquad u + v \leqslant p.$$

Then

$$(5.16) \quad E(a,\ b) + E'(up^s - a - 1,\ vp^s - b - 1) = s \quad (0 \leqslant a < up^s;\ 0 \leqslant b < vp^s).$$

Similarly we can show that

$$(5.17) \qquad E(a,\ b) = E(a,\ p^s - a - b - 1) \qquad (0 \leqslant a + b < p^s),$$

and

$$E(a+b,\ p^s-a-1)=E(a+b,\ p^s-b-1) \qquad (0 \leqslant a+b<p^s). \tag{5.18}$$

Combining (5.15), (5.17), (5.18) we get

$$\begin{aligned} E'(p^s-a-1,\ p^s-b-1) &= E'(a+b,\ p^s-a-1) \\ &= E'(a+b,\ p^s-b-1) \qquad (0 \leqslant a+b<p^s). \end{aligned} \tag{5.19}$$

6. – We shall now discuss the sum functions

$$S_j(r;\ x,\ y)=\sum_{a=0}^{p^r-1} F_j(r;\ x,\ y), \tag{6.1}$$

$$S'_j(r;\ x,\ y)=\sum_{a=0}^{p^r-1} G_j(r;\ x,\ y). \tag{6.2}$$

To begin with, it is evident that

$$S_0(r;\ x,\ y)=\sum_{a_0+b_0<p} \cdots \sum_{a_{r-1}+b_{r-1}<p} x^a y^b,$$

where

$$a=a_0+a_1 p+\ldots+a_{r-1} p^{r-1},$$

$$b=b_0+b_1 p+\ldots+b_{r-1} p^{r-1}.$$

It therefore follows from (2.17) that

$$S_0(r;\ x,\ y)=\prod_{k=0}^{r-1} f_0(x^{p^k},\ y^{p^k}). \tag{6.3}$$

Similarly we find that

$$S'_0(r;\ x,\ y)=f_1(x,\ y) \prod_{k=1}^{r-1} f_0(x^{p^k},\ y^{p^k}). \tag{6.4}$$

We now make use of the recurrences (3.7), (3.8), (3.9). It follows from

$$F_j(a_0 + ap;\ x,\ y) =$$

$$= c_{a_0}(x,\ y)\ F_j(a;\ x^p,\ y^p) + (xy)^{a_0-1}\ c_{p-a_0-2}(x,\ y)\ G_{j-1}(a-1;\ x^p,\ y^p)$$

$$(0 \leqslant a_0 < p;\ 0 \leqslant a < p^{r-1})$$

that

$$S_j(r;\ x,\ y) =$$

$$= f_0(x,\ y)\ S_j(r-1;\ x^p,\ y^p) + g_0(x,\ y)\ [S'_{j-1}(r-1;\ x^p,\ y^p) - G_{j-1}(p^{r-1}-1;\ x^p, y^p)].$$

Since

$$G_{j-1}(p^{r-1}-1) = 0 \qquad (j < r)\ ,$$

we get

$$S_j(r;\ x,\ y) = f_0(x,\ y)\ S_j(r-1;\ x^p,\ y^p) + g_0(x,\ y)\ S'_{j-1}(r-1;\ x^p,\ y^p) \qquad (j < r)\ . \tag{6.5}$$

Since

$$G_r(p^r-1;\ x,\ y) = \sum_{a=0}^{p^r-1} x^a\, y^{p^r-a-1} = \prod_{k=0}^{r-1} f_0(x^{p^k},\ y^{p^k})\ ,$$

while

$$G_j(p^r-1;\ x,\ y) = 0 \qquad (0 \leqslant j < r)\ ,$$

it is evident that

$$S'_r(r;\ x,\ y) = \prod_{k=0}^{r-1} f_0(x^{p^k},\ y^{p^k}) = S_0(r;\ x,\ y)\ . \tag{6.6}$$

Next, using

$$G_j(a_0+ap;\ x,\ y) = c_{a_0}(x,\ y)\ F_j(a;\ x^p,\ y^p) + (xy)^{a_0+1}\ c_{p-a_0-2}(x,\ y)\ G_{j-1}(a-1;\ x^p, y^p)$$

$$(0 \leqslant a_0 < p-1; \qquad 0 \leqslant a < p^{r-1})$$

and

$$G_j(p-1 + ap;\ x,\ y) = c_{p-1}(x,\ y)\ G_{j-1}(a;\ x^p,\ y^p)$$

$$(0 \leqslant a < p^{r-1})\ ,$$

we get

(6.7)
$$S_j'(r;\ x,\ y) =$$
$$= S_j(r;\ x,\ y) - c_{p-1}(x,\ y)\, S_j(r-1;\ x^p,\ y^p) + c_{p-1}(x,\ y)\, S_{j-1}'(r-1;\ x^p,\ y^p)\,.$$

If we take $j = r$ in (6.5) and (6.1), we get

(6.8)
$$S_{r-1}(r;\ x,\ y) = g_0(x,\ y)\, S_{r-2}'(r-1;\ x^p,\ y^p)\,,$$

(6.9)
$$S_{r-1}'(r;\ x,\ y) = g_1(x,\ y)\, S_{r-2}'(r-1;\ x^p,\ y^p)\,,$$

respectively. Since

$$S_0'(1;\ x,\ y) = f_1(x,\ y)\,,$$

iteration of (6.9) gives

(6.10)
$$S_{r-1}'(r;\ x,\ y) = \prod_{k=1}^{r-2} g_1(x^{p^k},\ y^{p^k}) \cdot f_1(x^{p^{r-1}},\ y^{p^{r-1}})\,.$$

Hence, by (6.8),

(6.11)
$$S_{r-1}(r;\ x,\ y) = g_0(x,\ y) \prod_{k=0}^{r-2} g_1(x^{p^k},\ y^{p^k}) \cdot f_1(x^{p^{r-1}},\ y^{p^{r-1}})\,.$$

It follows from (6.5) and (6.7) that

(6.12)
$$g_0(x^p,\ y^p)\,[S_{j+1}(r+1;\ x,\ y) - f_0(x,\ y)\, S_{j+1}(r;\ x^p,\ y^p)] =$$
$$= g_0(x,\ y)\,[g_1(x^p,\ y^p)\, S_j(r;\ x^p,\ y^p)\ +$$
$$- c_{p-1}(x^p,\ y^p)\{f_0(x^p,\ y^p) + g_0(x^p,\ y^p)\}\, S_j(r-1;\ x^{p^2},\ y^{p^2})] \qquad (j < r)\,.$$

If we take $j = 1$ in (6.5) we get

$$S_1(r;\ x,\ y) = f_0(x,\ y)\, S_1(r-1;\ x^p,\ y^p) + g_0(x,\ y)\, S_0'(r-1;\ x^p,\ y^p)\,.$$

In view of (6.3) and (6.4), this gives

$$\frac{S_1(r;\ x,\ y)}{S_0(r;\ x,\ y)} - \frac{S_1(r-1;\ x^p,\ y^p)}{S_0(r-1;\ x^p,\ y^p)} = \frac{g_0(x,\ y)\ f_1(x^p,\ y^p)}{f_0(x,\ y)\ f_0(x^p,\ y^p)}\,,$$

so that

$$\frac{S_1(r;\ x,\ y)}{S_0(r;\ x,\ y)} = \sum_{k=0}^{r-2} \frac{g_0(x^{p^k},\ y^{p^k})\, f_1(x^{p^{k+1}},\ y^{p^{k+1}})}{f_0(x^{p^k},\ y^{p^k})\, f_0(x^{p^{k+1}},\ y^{p^{k+1}})} \qquad (r > 1)\,. \tag{6.13}$$

We remark that (5.17) implies

$$\begin{aligned} S_j(r;\ x,\ y) &= y^{p^r - 1}\, S_j(r;\ xu^{-1},\ y^{-1}) \\ &= x^{p^r - 1}\, S_j(r;\ x^{-1},\ x^{-1} y)\,. \end{aligned} \tag{6.14}$$

References.

[1] L. Carlitz, *The number of binomial coefficients divisible by a fixed power of a prime*, Rend. Circ. Mat. Palermo (2) **16** (1967), 299-320.

Summary.

Let p be a fixed prime and let $\theta_j(n)$ denote the number of binomial coefficients

$$\binom{n}{k} \qquad (0 \leqslant k \leqslant n)$$

divisible by exactly p^j. Also let $\psi_j(n)$ denote the number of products

$$(n+1)\binom{n}{k} \qquad (0 \leqslant k \leqslant n)$$

divisible by exactly p^j. Numerous properties of $\theta_j(n)$ and $\psi_j(n)$ were obtained in a previous paper. In the present paper it is shown that most of these results can be extended to the functions $F_j(n;x,y)$, $G_j(n;x,y)$ defined by

$$F_j(n;\, x, y) = \sum_{\substack{a+b=n \\ E(a+b)=j}} x^a b^b, \qquad G_j(n;\, x, y) = \sum_{\substack{a+b=n \\ E'(a+b)=j}} x^a y^b,$$

where $E(a,b)$ denotes the greatest value of k such that

$$p^k \,\Big|\, \binom{a+b}{a}$$

and $E'(a,b)$ denotes the greatest value of k such that

$$p^k \,\Big|\, (a+b+1)\binom{a+b}{a}\,.$$

* * *